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Partition Functions of Matrix Models as the First Special Functions of String Theory II. Kontsevich Model

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In arXiv:hep-th/0310113 we started a program of creating a reference-book on matrix-model τ-functions, the new generation of special functions, which are going to play an important role in string theory calculations. The main focus of that paper was on the one-matrix Hermitian model τ-functions. The present paper is devoted to a direct counterpart for the Kontsevich and Generalized Kontsevich Model (GKM) τ-functions. We mostly focus on calculating resolvents (=loop operator averages) in the Kontsevich model, with a special emphasis on its simplest (Gaussian) phase, where exists a surprising integral formula, and the expressions for the resolvents in the genus zero and one are especially simple (in particular, we generalize the known genus zero result to genus one). We also discuss various features of generic phases of the Kontsevich model, in particular, a counterpart of the unambiguous Gaussian solution in the generic case, the solution called Dijkgraaf-Vafa (DV) solution. Further, we extend the results to the GKM and, in particular, discuss the p-q duality in terms of resolvents and corresponding Riemann surfaces in the example of dualities between (2,3) and (3,2) models.

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1 Introduction

1.1 Aim of the paper

In [1] a program was started to create a reference-book on matrix-model τ-functions – the new generation of special functions, which are going to play an important role in string theory calculations. The goal is to extract and considerably extend spectacular results obtained during the golden era of matrix model studies in late 80's and during sporadic moments of emerging new interest afterwards (see, e.g., reviews [2, 3] and references therein). In [4]-[6] a number of steps was made towards realization of this program for the *most fundamental* partition function of Hermitian matrix model. Integrable aspects of that theory were earlier considered in [7, 3]. Additional progress is made in the QFT-like approach, which is being developed in a complementary series of papers [8]. While we are still far from having a concise and exhaustive presentation in case of the Hermitian matrix model, time is also coming to extend analysis to other matrix-model τ-functions. The present paper being a sequel of [1] is a direct counterpart of [1] for the Kontsevich [9] and Generalized Kontsevich Model (GKM) [10, 11] τ-functions. For parallel consideration see [12]. Again, integrable and QFT aspects of the theory are mainly not included: we concentrate mostly on Virasoro-like constraints, perturbative resolvents (multi-densities) and duality properties, which are still insufficiently represented in the literature. In fact, the situation with the Kontsevich τ-function *per se* is a little better than with the Hermitian model, because its direct relation to the topological field theory [13] stimulated a relatively systematic consideration in the past, see [14]-[16] and references therein. Still, the most interesting part of the story – that of (p,q)-(q,p) duality – remained almost untouched after preliminary papers [17, 18, 19].

As soon as the topic can be hardly exhausted within one paper, here we concentrate only on a few basic examples leaving further developments for future publications. In particular, we mostly focus on calculating resolvents (=loop operator averages) in the Kontsevich model, with a special emphasis on its simplest (Gaussian) phase, where exists a surprising integral formula [16], and the expressions for the resolvents in genera zero and one are especially simple (in particular, we generalize the known genus zero result [20] to genus one, see also [21]). Thus, the program of [1] is realized for the Kontsevich model with several important simplifications: in variance with [1], there are very simple formulas for the n-point resolvent for lower genera; there is an integral formula for the Laplace transform of the n-point resolvent (summed over all genera!); and the (integral) equation for the generating function of all resolvents looks especially simple. In the paper, we give a review of these simplifications in sect.3, while before that, in sect.2, we discuss various features of generic phases of the Kontsevich model. Note that in a generic phase the main method to calculate resolvents is to solve the loop equations [22, 23, 24], which generally have many solutions, the Gaussian Kontsevich model being the only one that has the *unique* solution. However, there is a counterpart of this *unambiguous* Gaussian solution in the generic case, the solution called Dijkgraaf-Vafa (DV) solution [25]-[27]. Among other features, this solution has specific integrable properties. We discuss the DV solution in sect.4. The generalization of results to the Generalized Kontsevich model is contained in sect.5-6, where we also discuss the p-q duality in terms of resolvents and corresponding Riemann surfaces in the example of dualities between (2,3) and (3,2) models.

1.2 Correlators in matrix models

By essence, the main problem of matrix models one may address to is constructing a quantum field theory (QFT) presentation of matrix models. Solving this problem would allow one to effectively deal with all other problems.

The main purpose of QFT study of any model is to evaluate arbitrary correlation functions in an arbitrary phase and, after that, to study possible relations ("dualities") between these correlators in different phases. In the context of matrix models certain subsets of correlation functions are naturally collected into generating functions which will be called resolvents (or multi-densities). They possess, at least, three different representations.

La raison d'etre for (multi-)resolvents is a transparent group-theoretical structure of the Schwinger-

Dyson equations (which is obscure in a generic QFT but is immediately obvious in the simple matrix models): these are W- (Virasoro in the simplest cases) constraints with a loop-algebra structure. Accordingly, correlation functions satisfy the loop-equations, and the loop parameter becomes a natural expansion parameter of the generating functions.

It is still difficult to solve the genuine loop equation and obtain the *full* generating function, but an additional "genus expansion" converts the loop equation into a chain of simpler loop equations for *partial* generating functions, multi-resolvents which can be evaluated straightforwardly one after another. Ambiguities arising in this recursive process lead to different sets of multi-resolvents and are interpreted as associated with different phases of the theory. This will be our first approach to multi-resolvents.

Multi-resolvents emerge as non-trivial functions of the loop parameter z with singularities of various types, both poles and branchings. The second approach deals with them as poly-differentials on an auxiliary Riemann surface Σ_0 ("spectral" complex curve), and different phases correspond to different choices of the spectral surface and, in addition, to different conditions on the periods of multi-resolvent poly-differentials (e.g., if all the periods but the periods of the first multi-resolvent are vanishing, one gets to the so called Dijkgraaf-Vafa phase). As usual, poly-differentials on Riemann surfaces are most immediately represented as correlators of free fields, hence, this approach is often called conformal field theory (CFT) representation. An adequate reformulation of the loop equations suitable for the CFT representation is partly worked out in [8, 6].

The third approach represents correlation functions via (functional or matrix) integrals. The problem, however, is that the multi-resolvents are *generating functions* of matrix model correlators, i.e. derivatives of the matrix model partition function, and, therefore, their representation by integral formulas is not *a priori* obvious. In the Hermitian model this representation is rather straightforward: the spectral (or loop) parameter z is introduced through the average of the loop operator

$$\mathrm{Tr}\,\frac{dz}{z-\phi} = \sum_{k=0}^{\infty}\frac{dz}{z^{k+1}}\,\mathrm{Tr}\,\phi^k \longrightarrow \sum_{k=0}^{\infty}\frac{dz}{z^{k+1}}\frac{\partial}{\partial t_k} \tag{1.1}$$

where ϕ is the Hermitian matrix that is integrated over. However, a counterpart representation for the Kontsevich model remains unclear (see [18] for a very tedious approach to evaluating a few first $\frac{\partial}{\partial t_k}$ for the generalized Kontsevich integral). Worse than that, even integral formulas for the partition function are also unknown for most of non-trivial phases of the Kontsevich model. Still, spectacular results for correlators of the Gaussian Kontsevich model, due to [16] (discussed in sect.3), imply that the third approach should also be fruitful. Somewhat surprisingly the integral formulas in the Gaussian case are most simple not for the multi-resolvents themselves (i.e. not for the quantities subjected to the loop equations), but for their Laplace transforms.

1.3 IZK integral

Now we specify our general discussion to the case of the Kontsevich model, the main object of the present paper.

The story about the Kontsevich model begins from the Itzykson-Zuber-Kontsevich (IZK) integral over $n \times n$ Hermitian matrices X

$$I(\Lambda|V) = \frac{1}{\mathcal{N}(\Lambda)}\int dX\,\exp\Big(\mathrm{tr}\;\Lambda X \;-\; \mathrm{tr}\;V(X)\Big), \tag{1.2}$$

depending on the choice of the potential $V(x)$,

$$V(x) = \sum_{k=0}^{\infty} s_k x^k \tag{1.3}$$

and on the background matrix-valued field Λ. $\mathcal{N}(\Lambda)$ here is a normalization factor. This matrix model is actually of *the eigenvalue type* [7, 3]: as was first demonstrated by Itzykson and Zuber [28],

the integration over angular variables U in $X = U^+ X_{diag} U$ can be done explicitly, leaving the n-fold integral over eigenvalues χ_i of X in $X_{diag} = \mathrm{diag}(\chi_1, \dots, \chi_n)$,

$$I(\Lambda|V) \sim \prod_{i=1}^{n} \int d\chi_i e^{-V(\chi_i)} \frac{\Delta(\chi)}{\Delta(\lambda)} \det_{i,j=1,\dots,n} e^{\chi_i \lambda_j} \tag{1.4}$$

where λ_j are eigenvalues of Λ and Δ denotes the Van-der-Monde determinant,

$$\Delta(\chi) \equiv \det_{i,j=1,\dots,n} \chi_i^{j-1} = \prod_{i<j}^{n} (\chi_i - \chi_j)$$

Transition from (1.2) to (1.4) is typical for the Harish-Chandra-style character calculus in group theory [29, 30]. A particular expansion of the particular IZK integral with pure cubic potential $V(x) = x^3$ was related by M.Kontsevich [9] to cohomologies of the moduli space of Riemann surfaces and, finally [31], to partition function of topological gravity [13]. Moreover, this particular cubic potential case turns out to be related to more general Hodge integrals over the moduli space that include λ-classes [32], these latter being related to the Hurwitz numbers [33]. Many properties of the integral are, however, independent on particular choice of $V(x)$ and can be addressed in the theory of *Generalized Kontsevich Model* (GKM) [10, 34].

• The first split between different directions of study of the GKM concerns the type of Λ-dependence in (1.2). One option is to consider tr ΛX as a perturbation and represent $I(\Lambda)$ as a series in powers of tr Λ^k with $k > 0$ – this is the *character phase* of the model [30, 34]. Instead one can expand around a classical solution $X = L$ to the equation of motion $V'(L) = \Lambda$ of the full action, then the expansion will be in powers of $t_k = \frac{1}{k}\mathrm{tr} L^{-k}$ with $k > 0$ provided the normalization factor $\mathcal{N}$ in (1.2) is chosen equal to the quasiclassical value of the integral – this is the *Kontsevich phase* of the same model. In the GKM with *monomial* potential[1]

$$V_p(x) = \frac{x^{p+1}}{p+1}$$

L is just one of the p-th roots of Λ: $L^p = \Lambda$, moreover, in this case the integral does not depend on T_k with k divisible by p [10]. For non-monomial potentials $V(x)$ there are essentially different choices of L and essentially different Kontsevich phases (in their simplest phase, non-monomial potentials are reduced to the monomial ones, see [18, 50]).

• As usual for matrix models, the original integral (1.2) is not an adequate definition of the partition function: as it is, it describes reasonably only *some* of the phases. At the next step, it should be substituted with a set of differential equations w.r.t. the time variables t_k and s_k so that the partition function is defined to be a generic multi-branch solution to this system, with (1.2) providing integral representations for some of the branches. These equations have a simple form of *continuous Virasoro constraints* for the simplest case of $V(x) = \frac{1}{3}x^3$ [23, 24, 36, 31], i.e. for the original *Kontsevich model* [9], become more sophisticated $W^{(p+1)}$-constraints for the GKM with monomial potential [17, 24, 37, 38] and turn into even more sophisticated relations for a generic $V(x)$, especially when s-dependence is also taken into account. For fixed $V(x)$ the different branches in Kontsevich phase possess loop expansions and are further associated with shifts $t_k \to T_k + t_k$, so that expansions are in positive powers of t-variables with T appearing in denominators – just like in the case of the Hermitian matrix model. The (p, q)-model is the GKM with $V_p(x) = \frac{x^{p+1}}{p+1}$ in the phase with $T_k \neq 0$ for $k = 1, p+q$ only [23].

• One of the most remarkable properties of GKM is the *p-q duality* [17, 18, 19]: the relation between partition functions $Z_{p,q}$ and $Z_{q,p}$. It is not a literal coincidence between the two branches of the partition function, instead they are associated with two different coverings of one and the same spectral curve and should coincide after an appropriate change of time-variables.

[1] Note that p here can be negative equally well, the anti-polynomial Kontsevich case, [34].

• Since partition functions of all models, associated with the Itzykson-Zuber integral (1.2) possess the determinant representations (1.4), it is natural that they are τ-functions of the KP and Toda families in t-variables [10, 11, 7, 3]. They also possess certain integrability properties w.r.t. the s-variables [18, 11, 30, 50]. Thus, the entire theory of Itzykson-Zuber-Kontsevich models is indeed a piece of theory of stringy τ-functions. These τ-functions are, in fact, closely related to the Hermitian model τ-functions: both classes belong to the same matrix model M-theory [5].

• The main questions to be addressed in the course of study of every particular phase of every particular model are listed in the following table:

IZK integral =
partition function $I = \exp\left(\sum_p g^{2p-2}F^{(p)}\right)$

$\downarrow$

Ward identities in the form
of Virasoro and W constraints $\hat{L}(z)I = 0$

$\swarrow$ $\searrow$

genus-zero part $F^{(0)}$ of partition function — multi-resolvents $\rho^{(p\,|m)}(z_1, \ldots, z_m)$

$\downarrow$ $\downarrow$

bare spectral curve Σ_0 $\longrightarrow$ poly-differentials on Σ_0 $\downarrow$

$\downarrow$ $\downarrow$

full spectral curve Σ,
describing the matrix-model τ-function — Laplace transforms $\eta^{(p|m)}(x_1, \ldots, x_m)$

The bare spectral curve Σ_0 is an important characteristic of the branch of the partition function: different phases of the same model differ by the shape of Σ_0. In fact, in order to describe higher multi-resolvents in generic phases [1, 4], the spectral curve should be made dependent on the genus expansion parameter g, thus breaking the simple association between Σ_0 and $F^{(0)}$. The study of this phenomenon can be one of the clues to the construction of the last vertical arrow in the left column, relating Σ_0 with the full spectral curve Σ. As every KP/Toda τ-function, the partition function is formally associated with a point of the Grassmannian [39] and, thus, formally with some infinite genus Riemann surface: this is exactly what we call Σ. The horizontal line in the center of the table is a functorial map from complex curves to a hierarchical family of poly-differentials, which can be described and studied independently of other parts of the table. A big step in this direction is described in [12], but representation in terms of free fields on Σ_0 is still lacking, even in the simplest phases. Moreover, this map depends on additional conditions imposed on the poly-differentials, which are actually related to the choice between different branches of partition function, made also beyond the genus zero approximation. Also lacking is a description of the vertical arrow from poly-differentials to their Laplace transforms, which should be very interesting, because the Laplace transforms possess a very simple m-fold integral representation, at least, in some phases [16].

2 Kontsevich model

2.1 Solving loop equations

2.1.1 Kontsevich model: definitions

We start with defining the Kontsevich model. As explained in the Introduction, we define any matrix model partition function as a solution to an infinite set of equations. In particular, the Kontsevich

partition function is defined to satisfy the continuous Virasoro constraints:

$$\hat{L}_-(z)Z(t) = 0,$$

$$\hat{L}_-(z) = \Big(:\hat{J}^2(z):\Big)_- = \sum_n \frac{\hat{L}_n (dz)^2}{z^{n+2}} =$$

$$= \frac{g^2}{8}\sum_{n=-1}^{+\infty}\frac{(dz)^2}{z^{n+2}}\left(\sum_{k>0}(2k+1)\,t_{2k+1}\frac{\partial}{\partial t_{2(k+n)+1}} + \frac{g^2}{2}\sum_{a+b=n-1}\frac{\partial^2}{\partial t_{2a+1}\partial t_{2b+1}} + \frac{\delta_{n,0}}{8} + \frac{\delta_{n,-1}t_1^2}{2g^2}\right)$$

$$\hat{J}(z|t) = \frac{1}{4}\sum_{k=0}^{\infty}\left\{\Big(2k+1\Big)t_{2k+1}z^{k-1/2}dz + g^2\frac{dz}{z^{k+3/2}}\frac{\partial}{\partial t_{2k+1}}\right\} \tag{2.1}$$

In order to define the branch of the partition function, we shift the times,

$$t_{2k+1} = \tau_{2k+1} + T_{2k+1} \quad 0 \le k \le N, \tag{2.2}$$

and consider the partition function to be a formal power series in the shifted times τ_{2k+1} that satisfies (2.1).

Now we shall follow the line of paper [1] and rewrite (2.1) in the form of loop equations that admit recursion solving. To this end, we introduce

- the loop operator

$$\nabla(z) = \sum_{n\geqslant 0}\frac{1}{z^{n+3/2}}\frac{\partial}{\partial \tau_{2n+1}} \tag{2.3}$$

- the generating function for T_k (a polynomial of degree N)

$$W(z) = \sum_{k=0}^{N}(2k+1)z^{k-1/2}T_{2k+1} \tag{2.4}$$

- the generating function for τ_k (a power series)

$$v(z) = \sum_{k=0}^{\infty}(2k+1)z^{k-1/2}\tau_{2k+1} \tag{2.5}$$

- the projector onto the negative part of series

$$P_z^-\left\{\sum_{k=-\infty}^{+\infty} z^k a_k\right\} = \sum_{k\le -1} z^k a_k \qquad\qquad P_z^+ = 1 - P_z^- \tag{2.6}$$

- the free energy and its topological expansion w.r.t. to the genus p

$$Z(\tau) = e^{\frac{1}{g^2}\mathcal{F}(\tau)} \qquad\qquad \mathcal{F} = \sum_{p\geqslant 0} g^{2p}\mathcal{F}^{(p)} \qquad\qquad \mathcal{F}|_{\tau=0} = F[T] \tag{2.7}$$

- the generating resolvent

$$G(z|\tau) = \nabla(z)\mathcal{F}(\tau) \tag{2.8}$$

- and the multi-resolvents

$$\rho^{|m)}(z_1,\ldots,z_m) = \nabla(z_1)\cdots\nabla(z_m)\mathcal{F}|_{\tau=0} \tag{2.9}$$

$$\rho^{(p|m)}(z_1,\ldots,z_m) = \nabla(z_1)\cdots\nabla(z_m)\mathcal{F}^{(p)}|_{\tau=0} \tag{2.10}$$

- the set of f-functions generated by the R-check operator

$$P_z^-(W(z)G(z|\tau)) = W(z)G(z|\tau) - f(z|\tau) \tag{2.11}$$

$$f(z|\tau) = P_z^+(W(z)G(z|\tau)) = \check{R}(z)\mathcal{F}(t) \tag{2.12}$$

$$\check{R}(z) = \sum_{m=0}^{N-2}\sum_{k=m+2}^{N} z^{k-m-2}(2k+1)T_{2k+1}\frac{\partial}{\partial T_{2m+1}} \tag{2.13}$$

$$f^{(p|m+1)}(z|z_1,\ldots,z_m) = \nabla(z_1)\cdots\nabla(z_m)\check{R}(z)\mathcal{F}^{(p)}(\tau)|_{\tau=0} = \check{R}(z)\rho^{(p|m)}(z_1,\ldots,z_m) \tag{2.14}$$

2.1.2 The loop equation and recursion relations on the multi-resolvents

Now rewrite the Virasoro constraints (2.1) in the form of the loop equation for the resolvent

$$P_z^-(v(z)G(z)) + W(z)G(z) - f(z) + \frac{1}{2}G^2(z) + \frac{g^2}{2}\nabla(z)G(z) + \frac{g^2}{8z^2} + \frac{(\tau_1+T_1)^2}{2z} = 0 \tag{2.15}$$

Applying the operator ∇ to this equation k times, using the identity

$$\nabla(x)P_z^-\left\{v(z)h(z)\right\} = 2\partial_x\left\{\frac{\left(\frac{x}{z}\right)^{\frac{1}{2}}h(z) - h(x)}{z-x}\right\} + P_z^-\left\{v(z)\nabla(x)h(z)\right\} \tag{2.16}$$

and ultimately putting all $\tau_k = 0$, one comes to the set of recursion relations for the multi-resolvents,

$$\begin{aligned}
&2\sum_{i=1}^{k}\partial_{z_i}\left\{\frac{\left(\frac{z_i}{z}\right)^{\frac{1}{2}}\rho^{|k)}(z,z_1,\ldots,\hat{z}_i,\ldots,z_k) - \rho^{|k)}(z_1,\ldots,z_k)}{z-z_i}\right\} - f^{|k+1)}(z|z_1,\ldots,z_k)+\\
&+W(z)\rho^{|k+1)}(z,z_1,\ldots,z_k) + \frac{1}{2}\sum_{k_1+k_2=k}\rho^{|k_1+1)}(z,z_{i_1},\ldots,z_{i_{k_1}})\rho^{|k_2+1)}(z,z_{j_1},\ldots,z_{j_{k_2}})+\\
&\qquad+\frac{g^2}{2}\rho^{|k+2)}(z,z,z_1,\ldots,z_k) + \frac{g^2}{8z}\delta_{k,0} + \frac{1}{z\prod_{i=1}^{k}z_i^{3/2}}\frac{1}{(2-k)!}T_1^{2-k} = 0
\end{aligned} \tag{2.17}$$

These recursive relations are invariant with respect to two different scaling transformations with the following scaling exponents:

$$\begin{gathered}
\deg\tau_{2n+1} = n-1 \qquad \deg g^2 = \deg F = -3 \qquad \deg T_{2n+1} = n-1\\
\deg z = -1 \qquad \deg\nabla = \frac{5}{2} \qquad \deg\rho^{(p|k)} = -3 + \frac{5}{2}k + 3p
\end{gathered} \tag{2.18}$$

and

$$\begin{gathered}
\deg' g = \deg' t_i = 1 \qquad \deg' F = 2\\
\deg' z = 0 \qquad \deg'\nabla = -1 \qquad \deg'\rho^{(p|k)} = 2-k-2p
\end{gathered} \tag{2.19}$$

Making the genus expansion of the recursive equations (2.17), one obtains for the g^{2p}-term

$$\begin{aligned}
&2\sum_{i=1}^{k}\tilde{\partial}_{z_i}\left\{\frac{\left(\frac{z_i}{z}\right)^{\frac{1}{2}}\rho^{(p|k)}(z,z_1,\ldots,\hat{z}_i,\ldots,z_k) - \rho^{(p|k)}(z_1,\ldots,z_k)}{z-z_i}\right\} - f^{(p|k+1)}(z|z_1,\ldots,z_k)+\\
&+W(z)\rho^{(p|k+1)}(z,z_1,\ldots,z_k) + \frac{1}{2}\sum_{q=0}^{p}\sum_{k_1+k_2=k}\rho^{(q|k_1+1)}(z,z_{i_1},\ldots,z_{i_{k_1}})\rho^{(p-q|k_2+1)}(z,z_{j_1},\ldots,z_{j_{k_2}})+
\end{aligned}$$

$$+\frac{1}{2}\rho^{(p-1|k+2)}(z,z,z_1,\dots,z_k)+\frac{1}{8z}\delta_{k,0}\delta_{p,1}+\frac{1}{\prod_{i=1}^k z_i^{3/2}}\frac{1}{(2-k)!}T_1^{2-k}\delta_{p,0}=0 \tag{2.20}$$

These double recursion relations (in p and k) can be used to determine all the multi-resolvents $\rho^{(p|k+1)}$ recursively. E.g., for $\rho^{(0|1)}$ we have a quadratic equation

$$(\rho^{(0|1)}(z))^2+2W(z)\cdot\rho^{(0|1)}(z)-2\check{R}(z)F^{(0)}[T]+T_1^2/z \tag{2.21}$$

its solution being[2]

$$\rho^{(0|1)}(z)=-W(z)+y(z) \tag{2.22}$$

where $y(z)$ is a multi-valued function of z

$$y^2=(W(z))^2-\frac{T_1^2}{z}+2\check{R}(z)F^{(0)}[T] \tag{2.23}$$

Making further iterations, the multi-density $\rho^{(p|k+1)}$ enters (2.20) linearly with the factor $y(z)$, and one can make iterations, e.g., using some computer algebra system in the following order: $(0|1)\to(1|0)\to(0|2)\to(1|1)\to(2|0)\to(0|3)\to(1|2)\to\cdots$

Note that the recursion relations contain a lot of ambiguity encoded in the functions $f^{(p|1)}(z)=\check{R}(z)F^{(p)}[T]$. Indeed, $F[T]$ can be an arbitrary function that satisfies the two constraints $\check{L}_{-1}$, $\check{L}_0$ (these are L_{-1}- and L_0-constraints with all $\tau_k=0$):

$$\sum_{k=1}^N(2k+1)T_{2k+1}\frac{\partial F}{\partial T_{2k-1}}+\frac{T_1^2}{2}=0 \qquad\qquad \sum_{k=0}^N(2k+1)T_{2k+1}\frac{\partial F}{\partial T_{2k+1}}+\frac{g^2}{8}=0 \tag{2.24}$$

Therefore, the space of solutions to the loop equations (Virasoro constraints) is parameterized by such functions $F[T]$.

2.2 Solving the reduced Virasoro constraints

The general solution of the second equation of (2.24), i.e. of the L_0-constraint, is

$$F[T]=-\frac{g^2}{8(2N+1)}\log T_{2N+1}+\tilde{F}(\chi_1,\dots,\chi_N) \tag{2.25}$$

where

$$\chi_k=\frac{(2N-2k+1)!!}{(2N+1)!!}\cdot\frac{-2T_{2N-2k+1}}{(-2T_{2N+1})^{\frac{2N-2k+1}{2N+1}}} \tag{2.26}$$

and $\tilde{F}$ is an arbitrary function. Then the L_{-1}-constraint reads as

$$\sum_{k=0}^{N-1}\chi_k\frac{\partial\tilde{F}}{\partial\chi_{k+1}}=-\frac{[(2N+1)!!]^2}{8}\chi_N^2 \tag{2.27}$$

where $\chi_0=1$ is not an independent variable. Its general solution is

$$\tilde{F}=\frac{[(2N+1)!!]^2}{8}\int_0^{\eta_1}\chi_N^2(\tilde{\eta}_1,\eta_2\dots,\eta_N)d\tilde{\eta}_1+\tilde{\tilde{F}}(\eta_2,\dots,\eta_N) \tag{2.28}$$

where $\tilde{\tilde{F}}$ is a new arbitrary function, and we made the triangle change of variables[3] generated by the following relations between the generating functions

$$\chi(z)\equiv\sum_{k=1}^N\chi_k z^k \qquad \eta(z)\equiv\sum_{k=1}^N\eta_k z^k=\sum_{p=1}^\infty\frac{(-)^p}{p}\chi^p(z) \tag{2.29}$$

[2]Note that the non-meromorphic term $W(z)dz$ in $\rho^{(0|1)}dz$ just cancels the singular part of the z-expansion of ydz at infinity.

[3]Since this change of variables is triangle, the variables χ_* can be equally well expressed through η_*.

In other words, η_k for $k < N+1$ are defined from the equation $e^{\eta(z)} = \frac{1}{1+\chi(z)} + O(z^{N+1})$ with all χ_k and η_k equal to zero if $k > N$.

In order to prove that (2.28), indeed, solves (2.27), one suffices to note that, since

$$\eta_i = \oint_0 \sum_{p=1} \frac{(-)^p}{p} \frac{\chi^p(z)}{z^{i+1}} dz$$

the L_{-1}-constraint (2.27) reads as

$$\sum_{k=0}^{N-1} \chi_k \frac{\partial \tilde{F}}{\partial \chi_{k+1}} = \sum_{k=0}^{N-1} \sum_i \chi_k \frac{\partial \eta_i}{\partial \chi_{k+1}} \frac{\partial \tilde{F}}{\partial \eta_i} = \sum_i \sum_{k=0}^{N-1} \sum_{p=1} (-)^p \chi_k \frac{\partial \tilde{F}}{\partial \eta_i} \oint_0 \frac{\partial \chi(z)}{\partial \chi_{k+1}} \frac{\chi^{p-1}(z)}{z^{i+1}} dz =$$
$$= \sum_i \frac{\partial \tilde{F}}{\partial \eta_i} \sum_{p=1} (-)^p \oint_0 \frac{\chi^{p-1}(z)[1+\chi(z)]}{z^i} dz = -\sum_i \frac{\partial \tilde{F}}{\partial \eta_i} \oint_0 \frac{1}{z^i} dz = -\frac{\partial \tilde{F}}{\partial \eta_1} \tag{2.30}$$

Note that this choice of η-variables is in no way unique: e.g., one can equally well transform $\eta_2, \ldots, \eta_N$ to any new $N-2$ variables without changing the formulas above. For instance, one can request that the transformation is *linear* in all χ_l with $l \geq 3$,

$$\bar{\eta}_k = Q_k(\chi_1, \chi_2) + \sum_{l \geq 3}^{k} \chi_l P_{k,l}(\chi_1, \chi_2) \tag{2.31}$$

and check that there exist polynomials Q_k and $P_{k,l}$ that preserve relation (2.30). Inserting (2.31) into (2.30), one immediately obtains that these polynomials satisfy only the equations

$$\frac{\partial Q_k}{\partial \xi_1} = -\chi_2 P_{k,3} \qquad \frac{\partial P_{k,l}}{\partial \xi_1} = -P_{k,l+1} \tag{2.32}$$

where we changed $\chi_{1,2}$ for new variables $\xi_{1,2}$

$$\xi_1 \equiv \chi_1 \qquad \xi_2 \equiv \chi_2 - \frac{\chi_1^2}{2} \tag{2.33}$$

Equations (2.32) also have a lot of solutions. In particular, one can add to Q_k an arbitrary function of ξ_2. Choosing, e.g., this function to be zero and $P_{k,k} = 1$, one immediately obtains

$$P_{k,k-l} = (-)^l \frac{\xi_1^l}{l!} \qquad Q_k = (-)^k \left[\frac{\xi_1^{k-2}}{k-2} \xi_2 + \frac{\xi_1^k}{2k(k-3)!} \right] \tag{2.34}$$

2.3 Example of transition: $N = 2 \to N = 1$

Let us consider the limit of $T_5 \to 0$ and see how the space of solutions to the loops equations, which is parameterized by a function of one variable in the case of $N = 2$, reduces to the only solution in the case of $N = 1$.

For $N = 2$ one has

$$\chi_1 = \frac{1}{5} \frac{-2T_3}{(-2T_5)^{3/5}} \qquad \chi_2 = \frac{1}{15} \frac{-2T_1}{(-2T_5)^{1/5}} \tag{2.35}$$

$$\eta_1 = -\chi_1 \qquad \eta_2 = \frac{1}{2}\chi_1^2 - \chi_2 \tag{2.36}$$

$$F[T] = -\frac{g^2}{40} \log T_5 - 15^2 \left(\frac{1}{60}\chi_1^5 - \frac{1}{12}\chi_1^3\chi_2 + \frac{1}{8}\chi_1\chi_2^2 \right) + \tilde{\tilde{F}}(\eta_2) \tag{2.37}$$

In order to have a smooth transition as T_5 goes to zero, one has to cancel the singularity, i.e. $\tilde{\tilde{F}}(\eta_2)$ must have an asymptotics

$$\tilde{\tilde{F}}(\eta_2) = 15\sqrt{2} \cdot \eta_2^{5/2} - \frac{g^2}{48} \log \eta_2 + \mathcal{O}(1), \quad \eta_2 \to \infty \tag{2.38}$$

and, therefore, one obtains

$$F[T]^{(N=1)} = \lim_{T_5 \to 0} F[T]^{(N=2)} = -\frac{g^2}{24} \log T_1 - \frac{1}{18} \frac{T_1^3}{T_3} + \text{const} \tag{2.39}$$

which coincides with (3.2) below.

One may use the scaling symmetry (2.18) in order to further restrict the function $\tilde{\tilde{F}}$. Indeed, using that the scaling dimension of η_2 is $-6/5$, one immediately comes to the expansion

$$\tilde{\tilde{F}} = \sum_{p=0}^{\infty} g^{2p} C_p \eta_2^{5/2 \cdot (1-p)} + C_1' g^2 \log \eta_2 \tag{2.40}$$

if assuming that the symmetry (2.18) does not change the solution, or, putting this differently, that $\tilde{\tilde{F}}$ does not contain any additional dimensional parameters changing under this symmetry transformation. C_0 and C_1' can be obtained by comparing (2.40) with (2.38). Therefore, the asymptotic (2.38) corresponds actually to the semiclassical limit of $g \to 0$.

Geometrically this transition corresponds to degeneration of the torus into the sphere. Indeed, formula (2.23) describes the torus in the case of $N = 2$. As it follows from (2.37) and (2.38), $2\check{R}(z)F^{(0)}[T] = 10T_5 \partial F^{(0)}[T]/\partial T_1 \to 0$ as $T_5 \to 0$. Therefore, (2.23) transforms under this transition exactly into the sphere corresponding to $N = 1$.

2.4 Resolvents

As as explained above, starting from the one-point resolvent

$$\rho^{(0|1)}(z) = -W(z) + y(z) \tag{2.41}$$

one recursively calculates further resolvents:

$$\rho^{(0|2)}(x_1, x_2) = \frac{1}{y(x_1)} \left(2\partial_{x_2} \left[\frac{\left(\frac{x_2}{x_1}\right)^{\frac{1}{2}} \rho^{(0|1)}(x_1) - \rho^{(0|1)}(x_2)}{x_1 - x_2} \right] - f^{(0|2)}(x_1|x_2) + \frac{T_1}{x_1 x_2^{3/2}} \right) \tag{2.42}$$

$$\rho^{(1|1)}(x) = \frac{1}{y(x)} \left(\frac{1}{2} \rho^{(0|2)}(x, x) - f^{(1|1)}(x) + \frac{1}{8x} \right) \tag{2.43}$$

etc.

These resolvents can be given a geometric meaning. Indeed, the r.h.s. of (2.23) is actually a polynomial of degree $2N - 1$. Thus, equation (2.23) defines a hyperelliptic curve $\Sigma_0 = \mathcal{C}$ of genus $N - 1$ in a generic case. (This genus should not be confused with the genus corresponding to the expansion of the free energy in powers of g, usually labeled by p.) This bare spectral curve is actually essential for constructing all the multi-resolvents because these are meromorphic multi-differentials $\rho^{(p|m)} \equiv \rho^{(p|m)}(z_1, \cdots, z_m) dz_1 \cdots dz_m$ on this curve with specified singularities ($\rho^{(0|1)}$ and $\rho^{(0|2)}$ are distinguished differentials playing a specific role). Typically this leaves some room for adding holomorphic differentials which exactly corresponds to the ambiguity in solutions to the loop equations.

Indeed, because of equations (2.24), one has an arbitrary function F of $(N + 1) - 2 = N - 1$ variables, and, at each step of recursive computation of the multi-resolvents, one finds in $\rho^{(p|m)}$ some ambiguous terms $\sim \frac{\partial^m F^{(p)}}{\partial T_{2n_1+1} \cdots \partial T_{2n_m+1}}$, $0 \le n_i \le N - 2$. Fixing these terms is equivalent to fixing the

periods of $\rho^{(p|m)}$. All these terms are certainly fully determined by $\frac{\partial F^{(p)}}{\partial T_{2n+1}}$, i.e. by fixing the periods of $\rho^{(p|1)}(z)$.

As an example, consider the two-point resolvent. It can be rewritten in the form

$$\rho^{(0|2)} = \rho^{(0|2)}_{hol} + \rho^{(0|2)}_{glob} - \rho^{(0|2)}_{loc} \tag{2.44}$$

where $\rho^{(0|2)}_{hol}$ is a holomorphic bi-differential on the curve $\mathcal{C}$,

$$\rho^{(0|2)}_{hol}(x_1,x_2) = \frac{\check{K}(x_1,x_2)F^{(0)}[T]\,dx_1dx_2}{y(x_1)y(x_2)} \tag{2.45}$$

$\rho^{(0|2)}_{glob}$ is a meromorphic bi-differential on the curve $\mathcal{C}$ that has the singularity at $x_1 = x_2$ at the *both* sheets of the curve of the following type:

$$\rho^{(0|2)}_{glob} \sim \frac{2dx_1dx_2}{(x_1-x_2)^2} + O(1) \tag{2.46}$$

At last, $\rho^{(0|2)}_{loc} \equiv \rho^{(0|2)}_{glob}\Big|_{y(z)=\sqrt{z}}$ has the same behaviour as $\rho^{(0|2)}_{glob}$ at infinity and cancels the singularity (2.46) of $\rho^{(0|2)}_{glob}$ at one of the sheets of the curve.

Manifestly,

$$\rho^{(0|2)}_{loc} = \frac{[x_1+x_2]\,dx_1dx_2}{x_1^{1/2}x_2^{1/2}(x_1-x_2)^2} \tag{2.47}$$

$$\rho^{(0|2)}_{glob}(x_1,x_2) = \frac{1}{y(x_1)y(x_2)(x_1-x_2)^2}((x_1+x_2)B(x_1,x_2) + (x_1-x_2)^2C(x_1,x_2) + \check{A}(x_1,x_2)F^{(0)}[T]) \tag{2.48}$$

where

$$B(x_1,x_2) = \frac{W(x_1)W(x_2)}{\sqrt{x_1x_2}} + \left[\frac{x_1}{x_2} + \frac{x_2}{x_1} - 3\right]\frac{T_1^2}{x_1x_2}$$

$$C(x_1,x_2) = \left(T_1\left[\frac{W(x_2)}{\sqrt{x_2}} + \frac{W(x_1)}{\sqrt{x_1}}\right] + 3T_1T_3\right)/(x_1x_2)$$

$$\check{K}(x_1,x_2) = \sum_{m=0}^{N-2}\sum_{m'=0}^{N'-2}\sum_{k=m+2}^{N}\sum_{k'=m'+2}^{N} x_1^{k-m-2}x_2^{k'-m'-2}(2k+1)(2k'+1)T_{2k+1}T_{2k'+1}\frac{\partial}{\partial T_{2m+1}}\frac{\partial}{\partial T_{2m'+1}}$$

$$\check{A}(x_1,x_2) = -\sum_{k=4}^{N}\sum_{n=0}^{k-4}\left\{x_1^{k-m-3}[3x_2-5x_1+2n(x_2-x_1)] + (x_2\leftrightarrow x_1)\right\}(2k+1)T_{2k+1}\frac{\partial}{\partial T_{2n+1}} +$$
$$+4\left[5T_5\frac{\partial}{\partial T_1} + 7T_7\frac{\partial}{\partial T_1} + \frac{7}{2}(x_1+x_2)T_7\frac{\partial}{\partial T_1}\right]$$

Note that the numerator of $\rho^{(0|2)}_{glob}$ is actually a polynomial in x_1, x_2.

Since $\rho^{(0|2)}_{hol}$ is a holomorphic bi-differential, the second derivatives $\frac{\partial^2 F[T]}{\partial T_{2i+1}\partial T_{2j+1}}$ (entering this differential) control the periods of $\rho^{(0|2)}$ and do not affect its singularities. Note that the number of independent variables in $F[T]$ is equal to the genus of $\mathcal{C}$: $N-1$.

Similarly one can deal with other resolvents in order to check that the multi-resolvents $\rho^{(p|m)} = \rho^{(p|m)}(z_1,\cdots,z_m)dz_1\cdots dz_m$ are meromorphic multi-differentials (except for the cases (0|1) and (0|2)) on the curve $\mathcal{C}$ and generically have poles of order $6p+2m-4$ in points (and only in these point) where $y=0$. These multi-resolvents can be further restricted with using symmetries (2.18) and (2.19).

We discuss these general properties of multi-resolvents and its applications in more details in the next section in the simplest example of the Gaussian Kontsevich model.

2.5 CFT representation

As already mentioned in the Introduction, the two-point function $\rho^{(0|2)}_{glob}$ can be represented as a propagator in a certain CFT:

$$\rho^{(0|2)}_{glob}(z_1, z_2) = \langle \partial X(z_1) \partial X(z_2) \rangle \tag{2.49}$$

where X is some local field defined on $\mathbb{CP}^1$ parameterized by z. This is because there is a singularity at $z_1 = z_2$, i.e. when the arguments of the fields coincide. The CFT is defined by the covering $\mathcal{C} \xrightarrow{\pi} \mathbb{CP}^1$. One way is to consider the scalar field living on the $\mathcal{C}$ as a collection of two fields X_1, X_2 living on the corresponding sheets of the covering: $\mathcal{C} \xrightarrow{(X_1,X_2)} \mathbb{C}$. Then there will be a monodromy $X_1 \leftrightarrow X_2$ when z goes around a branch point $y = 0$. Then the field X is a linear combination of fields X_1, X_2 that diagonalizes this monodromy: $X = X_1 - X_2$, $X \leftrightarrow -X$.

To put this differently, let z parameterize the whole world-sheet now (i.e. topologically it would be a sphere), but the target space is now an orbifold: $\mathcal{C}/\mathbb{Z}_2 \simeq \mathbb{CP}^1 \xrightarrow{X} \mathbb{C}/\mathbb{Z}_2$. The branching points $y = 0$ are now just points where string wraps around the $\mathbb{Z}_2$-fixed point.

Both of these approaches can be actually described in the same way via the branching point operators of [40, 41]. To this end, one needs to introduce the twist field $\sigma_{1/2}(w, \bar{w})$ with the following operator product expansion (OPE):

$$\partial X(z) \sigma_{1/2}(w, \bar{w}) \sim (z - w)^{-1/2} \tau_{1/2}(w, \bar{w}) + \ldots \tag{2.50}$$

where $\tau_{1/2}$ is sometimes called excited twist field. Then, one can write

$$\langle \partial X(z_1) \partial X(z_2) \rangle = \left\langle \partial X(z_1) \partial X(z_2) \prod_{y(w_i)=0} \sigma_{1/2}(w_i, \bar{w}_i) \right\rangle_0 \tag{2.51}$$

By $\langle \cdot \rangle_0$ we denote the correlator in the ordinary CFT on $\mathbb{CP}^1$. This would provide us with the necessary structure of singularities

$$\langle \partial X(z_1) \partial X(z_2) \rangle \sim \frac{2}{(z_1 - z_2)^2} + O(1), \quad z_1 \to z_2 \tag{2.52}$$

$$\langle \partial X(z_1) \partial X(z_2) \rangle \sim (z_i - w_j)^{-1/2}, \quad z_i \to w_j, \quad y(w_j) = 0 \tag{2.53}$$

from which one can deduce (2.49).

One can also include $\rho^{(0|2)}_{hol}$ in this correlator. It would control the global monodromy properties of the filed X, i.e. how it changes when z goes around the cycles on $\mathcal{C}$.

At last,

$$\rho^{(0|2)}_{loc}(z_1, z_2) = \left\langle \partial X(z_1) \partial X(z_2) \sigma_{1/2}(0, 0) \right\rangle_0 \tag{2.54}$$

since $\rho^{(0|2)}_{loc}(z_1, z_2)$ knows nothing about the branching points w_i, see (2.47).

3 Gaussian branch of Kontsevich model

3.1 Specific of the Gaussian branch

In this section we consider the special, simplest case with only the first two times non-perturbatively turned on ($N = 1$). Then, the R-check operator (2.13) identically vanishes

$$\check{R}(z) = 0 \Rightarrow f^{(p|k)} = 0, \;\; \forall p, k \tag{3.1}$$

Therefore, there are no ambiguities in resolvents in this case. This key feature suggests a separate study of this distinguished case. This case is also a counterpart of the Gaussian branch of the Hermitian

matrix model, hence, we call it the Gaussian branch. To avoid misunderstanding, note that it has nothing to do with the Gaussian integral!

Given $T_1 = a$ and $T_3 = -\frac{1}{3}M$, the solution of (2.24) is

$$F[T] = F(M,a) = \frac{1}{6}\frac{a^3}{M} - \frac{g^2}{24}\log\frac{M}{M_0} \tag{3.2}$$

$$Z(M,a)|_{\tau=0} = \left(\frac{M}{M_0}\right)^{-\frac{1}{24}} e^{\frac{1}{6g^2}\frac{a^3}{M}} \qquad (Z|_{t_{2k+1}=-\frac{1}{3}M_0\delta_{k,1}} = 1) \tag{3.3}$$

and the curve is

$$y^2 = M^2(z-s) \qquad\qquad \left(s = \frac{2a}{M}\right) \tag{3.4}$$

$$W(z) = \frac{a}{\sqrt{z}} - M\sqrt{z} = \frac{M(z-s/2)}{\sqrt{z}} \tag{3.5}$$

Therefore, the resolvents non-trivially depend only on one parameter s (M can be effectively removed by rescalings). To simplify formulas, we consider from now on the redefined curve

$$Y(z) \equiv y(z)/M = \sqrt{z-s} \tag{3.6}$$

In the Gaussian case, the recurrent relation (2.20) can be simplified. More concretely, for sufficiently large indices it can be written in the form

$$y(z)\rho^{(p|k+1)}(z,z_1,\dots,z_k) = -2\sum_{i=1}^{k}\partial_{z_i}\frac{\rho^{(p|k)}(z_1,\dots,z_k)}{z-z_i} +$$
$$+\frac{1}{2}\sum_{\substack{k_1+k_2=k\\ p_1+p_2=p\\ k_i+p_i>0}}\rho_{\mathrm{mer}}^{(p_1|k_1+1)}(z,z_{i_1},\dots,z_{i_{k_1}})\rho_{\mathrm{mer}}^{(p_2|k_2+1)}(z,z_{j_1},\dots,z_{j_{k_2}}) + \frac{1}{2}\rho^{(p-1|k+2)}(z,z,z_1,\dots,z_k) \tag{3.7}$$

where the subscript $(\cdot)_{\mathrm{mer}}$ means that one has to replace $\rho^{(0|2)}$ with $\rho^{(0|2)}_{glob}$ leaving all other ρ's unchanged.

Indeed, for large enough indices (2.20) contains, in the Gaussian case, only four terms. The only term containing $\rho^{(0|1)}$ combines with that containing $W(z)$ to produce $y(z)$, while $\rho^{(0|2)}_{loc}$ in the sum quadratic in ρ's cancels the non-meromorphic ($\sim \left(\frac{z_i}{z}\right)^{\frac{1}{2}}$) part of the first term.

Recurrent relations (3.7) celebrate an important property that leads to drastic simplifications in the Gaussian case, which allows one to get rid of the only parameter s:

Important formula: *All the s-dependence of the resolvents (except for $p=0$, $m=1$ and $p=0$, $m=2$ cases) is actually encoded only in the differences $z_i - s$:*

$$\rho^{(p|m)}(z_1,\dots,z_m|s) = \rho^{(p|m)}(z_1-s,\dots,z_m-s|0) \tag{3.8}$$

In order to prove this formula, let us denote though l_{-1} the first-order part of the differential operator L_{-1}: $l_{-1} = \sum\limits_{k=1}^{\infty}(2k+1)t_{2k+1}\frac{\partial}{\partial t_{2k-1}}$, $L_{-1} = l_{-1} + \frac{t_1^2}{2g^2}$. Then, the L_{-1}-constraint on $\mathcal{F}$ is

$$l_{-1}\mathcal{F} = -\frac{t_1^2}{2} \tag{3.9}$$

To prove (3.8), one suffices to note that

$$[\,l_{-1},\nabla(z)] = 2\frac{\partial}{\partial z}\nabla(z) \qquad \text{and} \qquad l_{-1}|_{\tau=0} = 3\,T_3\frac{\partial}{\partial T_1} = -2\frac{\partial}{\partial s} \tag{3.10}$$

and, using these formulae, to show immediately that $-\frac{\partial}{\partial s}$ and $\sum_i \frac{\partial}{\partial z_i}$ acting on

$$\rho^{(p|m)}(z_1,\ldots,z_m) \stackrel{def}{=} \nabla(z_1)\cdots\nabla(z_m)\mathcal{F}^{(p)}|_{\tau=0} \tag{3.11}$$

are equal to each other whenever $(p,m) \neq (0,1)$ or $(0,2)$ (so that one can ignore the r.h.s. of (3.9)).

Formula (3.8) can be also proved by induction using the recursive relations (3.7). Indeed, the claim is correct for $\rho^{(0|2)}_{\mathrm{mer}}$ by an immediate check. Further, if all the ρ's in the r.h.s. of (3.7) enjoy the property (3.8) (by the induction assumption), this is also true for $z - z_i$ and $y(z)$ and, thus, for $\rho^{(p|k+1)}$ in the l.h.s.

We can use Y as a standard coordinate on our $\mathbb{CP}^1$, since $z = Y^2 + s$. Then (3.8) says that the densities can be written in terms of Y's only. Thus, the case of arbitrary s is, in a sense, equivalent to the $s = 0$ case.

3.2 Resolvents

In the next subsection, we present for a reference manifest expressions for several first densities. They all can be obtained recursively using (3.7), by hands or with the help of computer (using, e.g., MAPLE).

3.2.1 First resolvents

The one-point resolvents:

- Genus $p = 0$

$$\rho^{(0|1)}(z|s) = M\left(\sqrt{z} - \frac{s}{2\sqrt{z}} - Y(z)\right) = M\frac{s}{2}\sum_{n=1}^{\infty}\frac{s^n}{z^{n+1/2}}\frac{\Gamma(n+1/2)}{(n+1)!\,\Gamma(1/2)} \tag{3.12}$$

- Genus $p = 1$

$$\rho^{(1|1)}(z|s) = \frac{1}{8}\frac{1}{M}\frac{1}{Y^5(z)} \tag{3.13}$$

- Genus $p = 2$

$$\rho^{(2|1)}(z|s) = \frac{105}{128}\frac{1}{M^3}\frac{1}{Y^{11}(z)} \tag{3.14}$$

- Genus $p = 3$

$$\rho^{(3|1)}(z|s) = \frac{25025}{1024}\frac{1}{M^5}\frac{1}{Y^{17}(z)} \tag{3.15}$$

- Genus $p = 4$

$$\rho^{(4|1)}(z|s) = \frac{56581525}{32768}\frac{1}{M^7}\frac{1}{Y^{23}(z)} \tag{3.16}$$

The list of resolvents grouped by the genus p (we put $s = 0$ in all resolvents but $\rho^{(0|1)}$ and $\rho^{(0|2)}$; the s-dependence can be easily restored using formula (3.8)):

- Genus $p = 0$

$$\rho^{(0|1)}(z|s) = M\left(-\sqrt{z} + \frac{s}{2\sqrt{z}} - Y(z)\right) \tag{3.17}$$

$$\rho^{(0|2)}(z_1,\, z_2|s) = \frac{z_1 + z_2 - 2s}{(z_1 - z_2)^2 Y(z_1)Y(z_2)} - \frac{z_1 + z_2}{z_1^{1/2} z_2^{1/2}(z_1 - z_2)^2} \tag{3.18}$$

$$\left(\rho^{(0|2)}(z,z|s) = \frac{(z - s/2)(s/2)}{Y^2(z)}\right) \tag{3.19}$$

$$\rho^{(0|3)}(z_1,\, z_2,\, z_3|s) = \frac{1}{M}\,\frac{1}{Y^3(z_1)Y^3(z_2)Y^3(z_3)} \tag{3.20}$$

$$\rho^{(0|4)}(z_1,\, z_2,\, z_3,\, z_4|s=0) = \frac{3}{M^2}\frac{1}{z_1^{5/2}z_2^{5/2}z_3^{5/2}z_4^{5/2}}\cdot(z_1\,z_2\,z_3 + z_1\,z_2\,z_4 + z_1\,z_3\,z_4 + z_2\,z_3\,z_4) \tag{3.21}$$

$$\begin{aligned}\rho^{(0|5)}(z_1, z_2, z_3, z_4, z_5|0) = {}& \frac{3}{M^3\,{z_1}^{7/2}{z_2}^{7/2}{z_3}^{7/2}{z_4}^{7/2}{z_5}^{7/2}}\,(6\,{z_1}^2z_5{z_4}^2{z_3}^2z_2 + 5\,{z_1}^2{z_2}^2{z_5}^2{z_3}^2\\ &+ 5\,{z_1}^2{z_2}^2{z_3}^2{z_4}^2 + 6\,{z_1}^2{z_2}^2{z_5}^2z_4z_3 + 6\,{z_1}^2z_2{z_5}^2{z_4}^2z_3 + 6\,{z_1}^2{z_5}^2z_4{z_3}^2z_2 + 5\,{z_1}^2{z_5}^2{z_4}^2{z_3}^2\\ &+6\,{z_1}^2{z_2}^2{z_3}^2z_4z_5 + 6\,{z_1}^2{z_2}^2z_5{z_4}^2z_3 + 5\,{z_1}^2{z_2}^2{z_5}^2{z_4}^2 + 6\,z_1{z_2}^2z_5{z_4}^2{z_3}^2 + 6\,z_1z_2{z_5}^2{z_4}^2{z_3}^2\\ &+6\,z_1{z_2}^2{z_5}^2z_4{z_3}^2 + 6\,z_1{z_5}^2{z_4}^2z_3{z_2}^2 + 5\,{z_2}^2{z_5}^2{z_3}^2{z_4}^2)\end{aligned} \tag{3.22}$$

- Genus $p = 1$

$$\rho^{(1|1)}(z|s) = \frac{1}{8}\frac{1}{M}\frac{1}{Y^5(z)} \tag{3.23}$$

$$\rho^{(1|2)}(z_1,\, z_2|0) = \frac{1}{8M^2}\,\frac{5\,{z_1}^2 + 3\,z_1\,z_2 + 5\,{z_2}^2}{z_1^{7/2}z_2^{7/2}} \tag{3.24}$$

$$\begin{aligned}\rho^{(1|3)}(z_1,\, z_2,\, z_3|0) = {}& \frac{1}{8M^3}\frac{1}{z_1^{9/2}z_2^{9/2}z_3^{9/2}}\cdot(35\,{z_1}^3\,{z_2}^3 + 30\,{z_1}^3\,{z_2}^2\,z_3 + 30\,{z_1}^3\,z_2\,{z_3}^2 + 35\,{z_1}^3\,{z_3}^3\\ &+30\,{z_1}^2\,{z_2}^3\,z_3 + 18\,{z_1}^2\,{z_2}^2\,{z_3}^2 + 30\,{z_1}^2\,z_2\,{z_3}^3 + 30\,z_1\,{z_2}^3\,{z_3}^2 + 30\,z_1\,{z_2}^2\,{z_3}^3 + 35\,{z_2}^3\,{z_3}^3)\end{aligned} \tag{3.25}$$

$$\begin{aligned}\rho^{(1|4)}(z_1, z_2, z_3, z_4|0) = {}& \frac{3}{8\,M^4\,{z_1}^{11/2}{z_2}^{11/2}{z_3}^{11/2}{z_4}^{11/2}}(105\,{z_1}^4{z_2}^4z_3{z_4}^3\\ &+100\,{z_1}^4{z_2}^2{z_3}^4{z_4}^2 + 105\,{z_1}^4{z_4}^4{z_3}^4 + 90\,{z_1}^3{z_2}^2{z_3}^4{z_4}^3 + 105\,{z_3}^4{z_2}^3z_4{z_1}^4 + 54\,{z_3}^3{z_2}^3{z_4}^3{z_1}^3\\ &+100\,{z_1}^2{z_2}^2{z_3}^4{z_4}^4 + 90\,{z_3}^2{z_2}^3{z_4}^3{z_1}^4 + 90\,{z_3}^3{z_2}^3{z_4}^2{z_1}^4 + 100\,{z_1}^4{z_2}^2{z_3}^2{z_4}^4 + 90\,{z_2}^3{z_3}^2{z_4}^4{z_1}^3\\ &+105\,{z_3}^4{z_2}^4z_4{z_1}^3 + 90\,{z_2}^4{z_3}^3{z_4}^2{z_1}^3 + 105\,{z_3}^3{z_2}^4z_4{z_1}^4 + 90\,{z_3}^4{z_2}^3{z_4}^3{z_1}^2 + 90\,{z_3}^4{z_2}^3{z_4}^2{z_1}^3\\ &+90\,{z_1}^3{z_2}^2{z_3}^3{z_4}^4 + 105\,{z_1}^4{z_4}^4z_3{z_2}^3 + 90\,{z_1}^4{z_2}^2{z_3}^3{z_4}^3 + 90\,{z_2}^4{z_3}^3{z_4}^3{z_1}^2 + 105\,{z_2}^4{z_3}^3{z_4}^4z_1\\ &+90\,{z_2}^4{z_3}^2{z_4}^3{z_1}^3 + 100\,{z_2}^4{z_3}^2{z_4}^4{z_1}^2 + 100\,{z_2}^4{z_3}^4{z_4}^2{z_1}^2 + 100\,{z_2}^4{z_3}^2{z_4}^2{z_1}^4 + 105\,{z_1}^3{z_4}^4{z_3}^4z_2\\ &+105\,{z_1}^4{z_4}^3{z_3}^4z_2 + 105\,{z_2}^4{z_3}^4{z_4}^3z_1 + 105\,{z_2}^4{z_1}^4{z_3}^4 + 105\,{z_2}^3{z_3}^4{z_4}^4z_1 + 90\,{z_2}^3{z_3}^3{z_4}^4{z_1}^2\\ &+105\,{z_2}^4{z_3}^4{z_4}^4 + 105\,{z_1}^4{z_4}^4{z_3}^3z_2 + 105\,{z_2}^4{z_1}^4{z_4}^4 + 105\,{z_2}^4z_3{z_4}^4{z_1}^3)\end{aligned} \tag{3.26}$$

- Genus $p = 2$

$$\rho^{(2|1)}(z|s) = \frac{105}{128}\frac{1}{M^3}\frac{1}{Y^{11}(z)} \tag{3.27}$$

$$\rho^{(2|2)}(z_1,\ z_2|0) = \frac{35}{128 M^4} \frac{1}{z_1^{13/2} z_2^{13/2}} \cdot (33\, {z_1}^5 + 27\, {z_1}^4\, z_2$$
$$+ 29\, {z_1}^3\, {z_2}^2 + 29\, {z_1}^2\, {z_2}^3 + 27\, z_1\, {z_2}^4 + 33\, {z_2}^5)$$

$$\begin{aligned}\rho^{(2|3)}(z_1, z_2, z_3|0) = &\frac{35}{128\, M^5\, {z_1}^{15/2} {z_2}^{15/2} {z_3}^{15/2}} (396\, {z_1}^5 z_2 {z_3}^6 \\ &+ 396\, {z_2}^6 {z_3}^5 z_1 + 396\, {z_1}^2 {z_2}^6 {z_3}^4 + 406\, {z_2}^6 {z_1}^3 {z_3}^3 + 396\, {z_2}^6 {z_1}^4 {z_3}^2 + 396\, {z_2}^6 {z_1}^5 z_3 \\ &+ 396\, {z_2}^5 {z_1}^6 z_3 + 348\, {z_1}^4 {z_2}^5 {z_3}^3 + 324\, {z_1}^5 {z_2}^5 {z_3}^2 + 396\, {z_3}^6 {z_2}^5 z_1 + 324\, {z_3}^5 {z_2}^5 {z_1}^2 \\ &+ 348\, {z_3}^4 {z_2}^5 {z_1}^3 + 396\, {z_3}^6 {z_2}^4 {z_1}^2 + 429\, {z_2}^6 {z_3}^6 + 348\, {z_3}^5 {z_2}^4 {z_1}^3 + 396\, {z_1}^6 z_2 {z_3}^5 \\ &+ 396\, {z_2}^2 {z_1}^4 {z_3}^6 + 324\, {z_2}^2 {z_1}^5 {z_3}^5 + 406\, {z_1}^3 {z_2}^3 {z_3}^6 + 348\, {z_1}^4 {z_2}^3 {z_3}^5 + 429\, {z_1}^6 {z_3}^6 \\ &+ 348\, {z_1}^5 {z_3}^4 {z_2}^3 + 396\, {z_2}^2 {z_1}^6 {z_3}^4 + 360\, {z_2}^4 {z_1}^4 {z_3}^4 + 406\, {z_1}^6 {z_2}^3 {z_3}^3 \\ &+ 396\, {z_2}^4 {z_1}^6 {z_3}^2 + 348\, {z_2}^4 {z_1}^5 {z_3}^3 + 429\, {z_1}^6 {z_2}^6)\end{aligned} \tag{3.28}$$

- Genus $p = 3$

$$\rho^{(3|1)}(z|s) = \frac{25025}{1024} \frac{1}{M^5} \frac{1}{Y^{17}(z)} \tag{3.29}$$

$$\begin{aligned}\rho^{(3|2)}(z_1, z_2|0) = &\frac{35}{1024\, M^6\, {z_1}^{19/2} {z_2}^{19/2}} (12155\, {z_1}^8 + 10725\, {z_1}^7 z_2 + 11011\, {z_2}^2 {z_1}^6 + 11066\, {z_1}^5 {z_2}^3 \\ &+ 10926\, {z_1}^4 {z_2}^4 + 11066\, {z_2}^5 {z_1}^3 + 11011\, {z_1}^2 {z_2}^6 + 10725\, {z_2}^7 z_1 + 12155\, {z_2}^8)\end{aligned} \tag{3.30}$$

3.2.2 Resolvents: general formulae and relations

For generic p and m, $(p, m) \neq (0, 1)$ or $(0, 2)$, in the Gaussian case all the resolvents are of the following form:

$$\rho^{(p|m)}(z_1, \ldots, z_m|0) = \frac{1}{M^{2p+m-2}} \left(\frac{Q_{p,m}(\{z_i\})}{\prod_{i=1}^m Y^{6p+2m-3}(z_i)} \right) \tag{3.31}$$

where $Q_{p,m}$ are homogeneous symmetric polynomials in $\{z_i\}$ of degree $\deg Q_{p,m} = (m-1)(m+3p-3)$.

For $m = 1$ $\deg Q_{p,m} = 0$, and $Q_{p,m}$ is just a constant

$$\rho^{(p|1)}(z|s) = \frac{(6p-3)!!}{2^{3p} 3^p p!} \frac{1}{M^{2p-1}} \frac{1}{Y^{6p-1}(z)} \tag{3.32}$$

so that the formal power series $\rho^{|1)}(z) = \sum_{p=0}^{\infty} g^{2p} \rho^{(p|1)}(z)$ can be converted into the hypergeometric function. Its z-expansion is

$$\rho^{(p|1)}(z) = \frac{(6p-3)!!}{2^{3p} 3^p p!} \frac{1}{M^{2p-1}} \frac{1}{z^{3p-1/2}} \sum_{n=0}^{\infty} \frac{s^n}{z^n} \frac{\Gamma(n+3p-1/2)}{n!\, \Gamma(3p-1/2)} \tag{3.33}$$

which gives the general formula for the $\langle (\sigma_0)^n \sigma_{3p-2+n} \rangle$ intersection numbers [9].

There is also a general formula for the genus zero ($p = 0$) multi-resolvents:

$$\rho^{(0|k)}(\{z_i\}_{i=1}^k) = \frac{1}{M^{k-2}} (2\partial_s)^{k-3} \frac{1}{Y^3(z_1) \cdots Y^3(z_k)} \tag{3.34}$$

A general formula for the $p=1$ multi-resolvents is less explicit. It can be written with the help of the generating resolvent (2.8) treated as a functional of $\phi(z)$ such that $d\phi(z)/dz \equiv \frac{v(z)}{2}$, (2.5):

$$G^{(0)}[\phi] = \sum_{k=0}^{\infty} \frac{1}{M^k\, k!} \prod_{i=1}^{k} \oint_C dz_i \phi(z_i) \cdot\, (2\partial_s)^k \frac{1}{\prod_{j=1}^{k} Y^3(z_j)} = \tag{3.35}$$

$$= \sum_{k=0}^{\infty} \frac{(2\partial_s)^k}{M^k k!} \left(\oint_C \frac{\phi(z)dz}{(z^2-s)^{3/2}} \right)^k \Bigg|_{s=0} = \oint_0 e^{2t\partial_s} \frac{dt}{t - \frac{1}{M} \oint_C \frac{\phi(z)dz}{(z^2-s)^{3/2}}} \Bigg|_{s=0} = \oint_0 \frac{dt}{t - \frac{1}{M}\oint_C \mu(z)\phi(z)dz}$$

where $\mu(z) \equiv (z^2-2t)^{-3/2}$ and the contour C encircles ∞. Then, the logarithm of the generating resolvent generates the $p=1$ multi-resolvents

$$\frac{1}{24} \log G^{(0)}[\phi] = \sum_{k=0}^{\infty} \frac{1}{k!} \prod_{i=1}^{k} \left(\oint dz_i \phi(z_i) \right) \rho^{(1|k)}(z_1, \ldots, z_k), \tag{3.36}$$

i.e. the genus one multi-resolvents are connected parts (up to the factor of 24) of the k-point function generated by the generating resolvent,

$$\rho^{(1|k)}(z_1,\ldots,z_k) = \frac{1}{24} \frac{\delta \log G^{(0)}\{\phi\}}{\delta\phi(z_1)\ldots\delta\phi(z_k)} \Bigg|_{\phi=0}$$

For example,

$$\rho^{(1|1)}(z) = \frac{1}{24} \frac{\delta G^{(0)}\{\phi\}}{\delta\phi(z)} \Bigg|_{\phi=0} = \frac{1}{24M} \oint \frac{\mu(z)dt}{t^2} = \frac{1}{24M}\, \dot{\mu}(z)|_{t=0} = \frac{1}{8Mz^{5/2}}$$

$$\rho^{(1|2)}(z_1,z_2) = \frac{1}{24} \left(\frac{\delta^2 G^{(0)}\{\phi\}}{\delta\phi(z_1)\delta\phi(z_2)} - \frac{\delta G^{(0)}\{\phi\}}{\delta\phi(z_1)} \frac{\delta G^{(0)}\{\phi\}}{\delta\phi(z_2)} \right) \Bigg|_{\phi=0} =$$

$$= \frac{1}{24M^2} \left(2 \oint \frac{\mu(z_1)\mu(z_2)dt}{t^3} - \oint \frac{\mu(z_1)dt}{t^2} \oint \frac{\mu(z_2)dt}{t^2} \right) =$$

$$= \frac{1}{24M^2} \left(\frac{2\big(\ddot{\mu}(z_1)\mu(z_2) + 2\dot{\mu}(z_1)\dot{\mu}(z_2) + \mu(z_1)\ddot{\mu}(z_2)\big)}{2} - \dot{\mu}(z_1)\dot{\mu}(z_2) \right) \Bigg|_{t=0} = \frac{5z_1^2 + 3z_1z_2 + 5z_2^2}{8M^2 z_1^{7/2} z_2^{7/2}}$$

3.2.3 Proofs and comments

In the forthcoming considerations, we use the symmetries (2.18)[4] and (2.19) which should be supplemented with weights of genus-expanded multi-resolvents under the first transformation, (2.18): $\deg \rho^{(p|k)} = -3 + \frac{5}{2}k + 3p$, and under the second transformation, (2.19): $\deg' \rho^{(p|k)} = 2 - k - 2p$.

All the proofs in this subsection are done by induction. Note that we often omit as trivial checking the induction base. An equivalent way to obtain multi-resolvents is described in s.4.3.

[4]For $M=1,\ s=0$ this symmetry requirements are equivalent to the standard claim that the free energy has an expansion in τ such that

$$F(\tau) = \sum g^{2p} \tau_{2n_1+1} \cdots \tau_{2n_k+1} \cdot C^{(p)}_{n_1\cdots n_k}, \qquad \sum_{i=1}^{k} (n_i - 1) = 3p - 3 \tag{3.37}$$

C's being some *numerical* constants. This is usually derived [9] from the fact that the intersection number of a collection of forms is non-zero only when the sum of their degrees is equal to the dimension of the manifold.

Proof of (3.31): Using (3.7) it is easy to show by induction that $\rho^{(p|m)}(\{z_i\})$ is meromorphic on the $\mathbb{CP}^1$ in each z_i and can have poles only at $Y=0$. Then, in order to prove (3.31), it is enough to show (due to the symmetricity in $\{z_i\}$) that for some z_j the order of pole in $Y=0$ is $(6p+2m-3)$. This also can be done straightforwardly by induction. Then the degree of the polynomial $Q_{p,m}$ is completely fixed by symmetry (2.18): $\deg Q_{p,m} = (m-1)(m+3p-3)$, while the power of M in (3.31) can be determined by symmetry (2.19). □

In fact, the second symmetry (2.19) allows us to put hereafter $M=1$, while the first symmetry (2.18) does not affect M at all.

Proof of (3.34): Without any loss of generality one can put $s=0$ (i.e. $y(z)=z^{1/2}$) using formula (3.8) and identify $\partial_s \equiv -\sum_{i=0}^{k} \partial_{z_i}$. From the definition of ρ's and using symmetry (2.18), one obtains

$$\rho^{(0|k)}(\{z_i\}_{i=1}^{k}) = \sum_{\sum n_i = k-3} \frac{C_{n_1 \ldots n_k}}{\prod_{i=1}^{k} z^{3/2+n_i}} \tag{3.38}$$

In each term of the sum at the r.h.s. of this expression there exists i such that $n_i=0$. Let us denote through the subscript A the coefficient in front of $\frac{1}{z^{3/2}}$ in the asymptotic expansion at $z \to \infty$. Since ρ is symmetric in z_i's,

$$\rho_A^{(0|k)}(\{z_i\}_{i=1}^{k-1}) \equiv \sum_{\sum n_i = k-3} \frac{C_{n_1 \ldots n_{k-1} 0}}{\prod_{i=1}^{k-1} z^{3/2+n_i}} \tag{3.39}$$

contains the same information as $\rho^{(0|k)}$ itself. Thus, ρ can be in principle restored from ρ_A. However, we do not need to do this explicitly. Just assume (3.34) is correct for $k \le K$ (it is trivially correct for $K=2$). To prove it for $k=(K+1)$, it is enough to show that

$$\rho_A^{(0|K+1)}(\{z_i\}_{i=1}^{K}) = \left(-2\sum_{i=1}^{K} \partial_i\right)^{K-2} \frac{1}{\prod_{i=1}^{K} z_i^{3/2}} \tag{3.40}$$

One can easily do it by considering $\sim \frac{1}{z}$ term of the asymptotic of the recursive relation (3.7):

$$\rho_A^{(0|K+1)}(\{z_i\}_{i=1}^{K}) = \left(-2\sum_{i=1}^{K} \partial_i\right) \rho^{(0|K)}(\{z_i\}_{i=1}^{K}) \tag{3.41}$$

Then, (3.34) is correct by induction. □

Proof of (3.36): Similarly to the genus zero case, put $s=0$ and write

$$\rho^{(1|k)}(\{z_i\}_{i=1}^{k}) = \sum_{\sum n_i = k} \frac{C_{n_1 \ldots n_k}}{\prod_{i=1}^{k} z_i^{3/2+n_i}} \tag{3.42}$$

Now either there exists such i that $n_i=0$ or $\forall i \; n_i=1$. Therefore, in contrast with the genus zero case, all the information about $\rho^{(1|k)}$ is contained both in

$$\rho_A^{(1|k)}(\{z_i\}_{i=1}^{k-1}) \equiv \sum_{\sum n_i = k} \frac{C_{n_1 \ldots n_{k-1} 0}}{\prod_{i=1}^{k-1} z_i^{3/2+n_i}} \tag{3.43}$$

and in $c_k \equiv 3^{-k} C_{1 \ldots 1}$. First, we deal with this c_k. One can easily construct such ϕ that $\oint_C dx \frac{\phi(x)}{x^{n+5/2}} = \frac{\alpha}{3}\delta_{n,0}$. Now, we are interested only in the terms where each ∂_s acts once on each $\frac{1}{y(z_i)^3}$ in the k-point function in (3.35). Then, (3.35) reads as

$$G[\phi] = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \cdot k! = \frac{1}{1-\alpha} \tag{3.44}$$

and

$$\log G[\phi] = -\log(1-\alpha) = \sum_{k=1}^{\infty} \frac{\alpha^k}{k} \tag{3.45}$$

i.e., (3.36) is equivalent to the equality

$$c_k = \frac{(k-1)!}{24} \tag{3.46}$$

Now, as usual, we prove it by induction. Assume that, indeed, $c_k = (k-1)!/24$ for $k \leq K$. Computing the term $\frac{1}{z^2 \prod_{i=1}^{K} z_i^{3/2}}$ of the asymptotic of (3.7), one obtains (again only the first term at the r.h.s. survives)

$$3^{K+1} c_{K+1} = -2K(3^K c_K) + 5K(3^k c_K) = 3^{K+1} K c_K \tag{3.47}$$

where the two terms come from differentiating the denominator $(z - z_i)$ and the numerator respectively (one then expands $\frac{1}{z-z_i}\partial_{z_i} = \frac{1}{z}\partial_{z_i} + \frac{1}{z^2} z_i \partial_{z_i} + \ldots$). Therefore, $c_K = \frac{(K+1)!}{24}$.

As the second step, we study $\rho_A^{(1|k)}$. The $\frac{1}{z}$-asymptotic of the (3.7) gives us

$$\rho_A^{(1|k+1)}(\{z_i\}_{i=1}^k) = \left(-2\sum_{i=1}^{k} \partial_i\right) \rho^{(1|k)}(\{z_i\}_{i=1}^k). \tag{3.48}$$

Introduce the notation $\tilde{\rho}^{(k)}(\{z_i\}_{i=1}^k) \equiv (2\partial_s)^k \frac{1}{\prod_{j=1}^k y^3(z_j)}$ and $\tilde{\rho}^{(k)}_{\text{conn}}(\{z_i\}_{i=1}^k)$ for its connected part. Now we again apply the induction and assume that, for $k \leq K$, $\rho^{(1|k)}(\cdot) = \frac{1}{24}\tilde{\rho}^{(k)}_{\text{conn}}(\cdot)$. To prove it for $k = K+1$ it is enough to show that

$$\rho_A^{(1|K+1)}(\cdot) = \frac{1}{24}\tilde{\rho}^{(K+1)}_{\text{conn}, A}(\cdot) \tag{3.49}$$

Due to (3.48) and to the induction assumption, it is equivalent to

$$\tilde{\rho}^{(K+1)}_{\text{conn}, A}(\cdot) = 2\partial_s \tilde{\rho}^{(K)}_{\text{conn}}(\cdot) \tag{3.50}$$

For the complete functions such a relation is obvious from the definition: $\tilde{\rho}_A^{(K+1)}(\cdot) = 2\partial_s \tilde{\rho}^{(K)}(\cdot)$. Let $I = 1 \ldots k$ and $z_0 \equiv z$. The complete $(K+1)$-point function is expressed through the connected ones as follows

$$\begin{aligned} \tilde{\rho}^{(K+1)}(\{z_i\}_{i \in I \cup \{0\}}) &= \sum_{\bigsqcup_{j=1}^{s} \tilde{I}_j = I \cup \{0\}} \prod_{j=1}^{s} \tilde{\rho}^{(|\tilde{I}_j|)}_{\text{conn}}(\{z_i\}_{i \in \tilde{I}_j}) = \\ &= \sum_{\bigsqcup_{j=1}^{s} I_j = I} \sum_{l=1}^{s} \prod_{\substack{j=1 \\ j \neq l}}^{s} \tilde{\rho}^{(|I_j|)}_{\text{conn}}(\{z_i\}_{i \in \tilde{I}_j}) \tilde{\rho}^{(|I_l|+1)}_{\text{conn}}(\{z_i\}_{i \in I_l \cup \{0\}}) \end{aligned} \tag{3.51}$$

The $\frac{1}{z}$-asymptotic of this equation is

$$\tilde{\rho}_A^{(K+1)}(\{z_i\}_{i \in I}) = \sum_{\bigsqcup_{j=1}^{s} I_j = I} \sum_{l=1}^{s} \prod_{\substack{j=1 \\ j \neq l}}^{s} \tilde{\rho}^{(|I_j|)}_{\text{conn}}(\{z_i\}_{i \in \tilde{I}_j}) \cdot \tilde{\rho}^{(|I_l|+1)}_{\text{conn}, A}(\{z_i\}_{i \in I_l}) \tag{3.52}$$

Now, using the induction assumption,

$$2\partial_s \tilde{\rho}^{(K)}(\{z_i\}_{i \in I}) = \tilde{\rho}_{\text{conn}, A}(\{z_i\}_{i \in I}) + \sum_{\bigsqcup_{j=1}^{\mathbf{s>1}} I_j = I} \sum_{l=1}^{s} \prod_{\substack{j=1 \\ j \neq l}}^{s} \tilde{\rho}^{(|I_j|)}_{\text{conn}}(\{z_i\}_{i \in \tilde{I}_j}) \cdot 2\partial_s \tilde{\rho}^{(|I_l|)}_{\text{conn}}(\{z_i\}_{i \in I_l}) \tag{3.53}$$

$$\Leftrightarrow \qquad 2\partial_s \sum_{\bigsqcup_{j=1}^{s} I_j = I} \prod_{j=1}^{s} \tilde{\rho}^{(|I_j|)}_{\mathrm{conn}}(\{z_i\}_{i\in\tilde{I}_j}) = \tilde{\rho}_{\mathrm{conn},\,A}(\{z_i\}_{i\in I}) + 2\partial_s \sum_{\bigsqcup_{j=1}^{s>1} I_j = I} \prod_{j=1}^{s} \tilde{\rho}^{(|I_j|)}_{\mathrm{conn}}(\{z_i\}_{i\in\tilde{I}_j}) \tag{3.54}$$

$$\Longrightarrow \quad \tilde{\rho}^{(K+1)}_{\mathrm{conn},\,A}(\{z_i\}_{i\in I}) = 2\partial_s \tilde{\rho}^{(K)}_{\mathrm{conn}}(\{z_i\}_{i\in I}) \qquad \Longrightarrow \qquad \rho^{(1|K+1)}_{A}(\{z_i\}_{i\in I}) = \frac{1}{24}\tilde{\rho}^{(K+1)}_{\mathrm{conn},\,A}(\{z_i\}_{i\in I}) \tag{3.55}$$

□

3.3 Matrix integral representation

This partition function Z_K can be presented as the Hermitian matrix integral depending on the external matrix A,

$$Z_K = \frac{\int DX \ \exp\left(-\frac{g^2}{3}\mathrm{Tr}X^3 - \frac{g}{\sqrt{3}}\mathrm{Tr}AX^2\right)}{\int DX \ \exp\left(-\frac{g}{\sqrt{3}}\mathrm{Tr}AX^2\right)} \tag{3.56}$$

where the integral is understood as a perturbative power series in $\tau_{2k+1} \equiv g\dfrac{3^{2k+1}}{2k+1}\mathrm{Tr}A^{-2k-1}$. Note that this integral does not depend on the size of matrices X and A provided it is being considered as a function of t_k [10]. By the shift of the integration variable, it can be also reduced to the form

$$Z_K = \exp\left(-\frac{2}{3g}\mathrm{Tr}\Lambda^{\frac{3}{2}}\right)\frac{\int DX \ \exp\left(-\frac{g^2}{3}\mathrm{Tr}X^3 + \mathrm{Tr}\Lambda X\right)}{\int DX \ \exp\left(-\frac{g}{\sqrt{3}}\mathrm{Tr}AX^2\right)} \tag{3.57}$$

where $3\Lambda = A^2$, i.e. $\tau_{2k+1} \equiv g\dfrac{1}{2k+1}\,\mathrm{Tr}\,\Lambda^{-k-\frac{1}{2}}$.[5]

3.4 Okounkov's representation of the Laplace transformed resolvents

Further on in this section we put $M = 2$, $g = 1$ and $s = 0$.

3.4.1 Laplace transform of the resolvents

Let us introduce the Laplace-transformed resolvents η:

$$\rho^{|k)}(z_1,\ldots,z_k) = 2^k \int_0^\infty \prod_{i=1}^{k} dx_i e^{-\sum_i x_i z_i}\eta^{|k)}(x_1,\ldots,x_k) \tag{3.58}$$

$$\eta^{|k)}(x_1,\ldots,x_k) = \frac{1}{(4\pi\, i)^k}\oint_C \prod_{i=1}^{k} dz_i e^{\sum_i x_i z_i}\rho^{|k)}(z_1,\ldots,z_k) \tag{3.59}$$

where the contour C encircles 0, beginning and ending at the negative infinity with respect to the branch cut along the negative real ray.

The manifest expressions for a few first η are

$$\eta^{(0|3)}(x_1,x_2,x_3) = \frac{1}{2\,\pi^{3/2}}\, x_1^{1/2}x_2^{1/2}x_3^{1/2} \tag{3.60}$$

---

[5] In order to introduce an arbitrary shifted first time, $t_{2k+1} = \tau_{2k+1} - \frac{M}{3}\delta_{k,1}$, where M is a parameter, one should consider instead of (3.56) the integral

$$Z_K = \frac{\int DX \ \exp\left(-\frac{16g^2}{3M^2}\mathrm{Tr}X^3 - \frac{2\sqrt{2}g}{\sqrt{3M}}\mathrm{Tr}AX^2\right)}{\int DX \ \exp\left(-\frac{2\sqrt{2}g}{\sqrt{3M}}\mathrm{Tr}AX^2\right)} = \exp\left(-\frac{M}{6g}\mathrm{Tr}\Lambda^{\frac{3}{2}}\right)\frac{\int DX \ \exp\left(-\frac{16g^2}{3M^2}\mathrm{Tr}X^3 + \mathrm{Tr}\Lambda X\right)}{\int DX \ \exp\left(-\frac{2\sqrt{2}g}{\sqrt{3M}}\mathrm{Tr}AX^2\right)}$$

which is a function of the same $\tau_{2k+1} = g\dfrac{3^{k+\frac{1}{2}}}{k+\frac{1}{2}}\mathrm{Tr}A^{-2k-1} = \dfrac{g}{k+\frac{1}{2}}\mathrm{Tr}\Lambda^{-k-\frac{1}{2}}$.

$$\eta^{(0|4)}(x_1,x_2,x_3,x_4) = \frac{1}{2\,\pi^2}\,(x_1^{1/2}x_2^{1/2}x_3^{1/2}x_4^{3/2} + x_1^{1/2}x_2^{1/2}x_3^{3/2}x_4^{1/2} + x_1^{1/2}x_2^{3/2}x_3^{1/2}x_4^{1/2} + x_1^{3/2}x_2^{1/2}x_3^{1/2}x_4^{1/2}) \tag{3.61}$$

$$\eta^{(1|1)}(x_1) = \frac{1}{24\,\pi^{1/2}}\,x_1^{3/2} \tag{3.62}$$

$$\eta^{(1|2)}(x_1,x_2) = \frac{1}{24\,\pi}\,(x_1^{1/2}x_2^{5/2} + x_1^{5/2}x_2^{1/2} + x_1^{3/2}x_2^{3/2}) \tag{3.63}$$

$$\begin{aligned}\eta^{(1|3)}(x_1,x_2,x_3) = \frac{1}{24\,\pi^{3/2}}\,(x_1^{1/2}x_2^{1/2}x_3^{7/2} + x_1^{1/2}x_2^{7/2}x_3^{1/2} + x_1^{7/2}x_2^{1/2}x_3^{1/2} + 2x_1^{1/2}x_2^{3/2}x_3^{5/2}\\ +2x_1^{1/2}x_2^{5/2}x_3^{3/2} + 2x_1^{3/2}x_2^{1/2}x_3^{5/2} + 2x_1^{5/2}x_2^{1/2}x_3^{3/2} + 2x_1^{3/2}x_2^{5/2}x_3^{1/2}\\ +2x_1^{5/2}x_2^{3/2}x_3^{1/2} + 2x_1^{3/2}x_2^{3/2}x_3^{3/2})\end{aligned} \tag{3.64}$$

$$\eta^{(2|1)}(x_1) = \frac{1}{2^{10}\,9\,\pi^{1/2}}\,x_1^{9/2} \tag{3.65}$$

$$\begin{aligned}\eta^{(2|2)}(x_1,x_2) = \frac{1}{2^6\,45\,\pi}(5x_1^{1/2}x_2^{11/2} + 5x_1^{11/2}x_2^{1/2} + 15x_1^{3/2}x_2^{9/2}\\ +15x_1^{9/2}x_2^{3/2} + 29x_1^{5/2}x_2^{7/2} + 29x_1^{7/2}x_2^{5/2})\end{aligned} \tag{3.66}$$

One can easily turn on nonzero s using $\eta^{(p|m)}(x_1,\ldots,x_m|s) = e^{s\sum_i x_i}\eta^{(p|m)}(x_1,\ldots,x_m|0)$ (except for $p=0$,$m=1$ and $p=0$,$m=2$ cases).

The general formula for the genus zero η-resolvents can be easily obtained from (3.34):

$$\begin{aligned}\eta^{(0|k)}(x_1,\ldots,x_k) = \frac{1}{(4\pi\,i)^k}\oint_C \prod_{i=1}^k dz_i e^{\sum_i x_i z_i}\frac{1}{2}\left(-\sum_{i=1}^k \partial_{z_i}\right)^{k-3}\frac{1}{z_1^{3/2}\ldots z_k^{3/2}} =\\ = \frac{\left(\sum_{i=1}^k x_i\right)^{k-3}}{2\,(4\pi\,i)^k}\oint_C \prod_{i=1}^k dz_i e^{\sum_i x_i z_i}\frac{1}{z_1^{3/2}\ldots z_k^{3/2}} = \frac{1}{2\,\pi^{k/2}}\left(\sum_{i=1}^k x_i\right)^{k-3}\prod_{i=1}^k x_i^{1/2}\end{aligned} \tag{3.67}$$

3.4.2 One point function

One can easily verify that the one-point η-resolvent is as follows

$$\eta^{|1)}(x) = \frac{1}{2\,\sqrt{\pi}}\,\frac{e^{\frac{x^3}{12}}}{x^{3/2}} \tag{3.68}$$

The r.h.s. of (3.58) is then equal to

$$\begin{aligned}2\int_0^\infty dx\,\eta(x)e^{-zx} = \frac{1}{\sqrt{\pi}}\sum_{p=0}^\infty \frac{1}{p!\,12^p}\int_0^\infty dx x^{3p-3/2}e^{-xz} =\\ = \frac{1}{\sqrt{\pi}}\sum_{p=0}^\infty \frac{1}{p!\,12^p}\,\frac{\Gamma(3p-1/2)}{z^{3p-1/2}} = \sum_{p=0}^\infty \frac{(6p-3)!!}{p!\,12^p\,2^{3p-1}}\,\frac{1}{z^{3p-1/2}}\end{aligned} \tag{3.69}$$

Exactly the same expression is obtained from (3.33)

$$\rho^{|1)} = \sum_{p=0}^\infty \rho^{(p|1)}(x) = \sum_{p=0}^\infty \frac{(6p-3)!!}{p!\,24^p\,2^{2p-1}}\,\frac{1}{z^{3p-1/2}} \tag{3.70}$$

3.4.3 Okounkov's result

In [16] Okounkov obtained a representation for the resolvents in terms of finite-dimensional integrals which can be thought of as a discrete version of a special functional integral. He introduced the functions

$$\mathcal{E}(x_1,\ldots,x_n) = \frac{1}{2^n \pi^{n/2}} \frac{\exp\left(\frac{1}{12}\sum x_i^3\right)}{\prod\sqrt{x_i}} \int_{s_i\geq 0} ds\, \exp\left(-\sum_{i=1}^{n}\frac{(s_i-s_{i+1})^2}{4x_i} - \sum_{i=1}^{n}\frac{s_i+s_{i+1}}{2}\,x_i\right) \quad (3.71)$$

and their symmetrized versions

$$\mathcal{E}^{\circlearrowleft}(x) = \sum_{\sigma\in S(n)/(12\ldots n)} \mathcal{E}(x_{\sigma(1)},\ldots,x_{\sigma(s)})\,, \quad (3.72)$$

where the summation is over coset representatives modulo the cyclic group generated by the permutation $(12\ldots n)$.

Then the resolvent is a generating function for intersection numbers on moduli spaces of curves with n fixed points (genus is arbitrary) and is equal (up to some renormalisation and rescaling) to the sum[6]

$$\eta^{|n)}(x_1,\ldots,x_n) = \sum_{\alpha\in\Pi_n} (-1)^{\ell(\alpha)+1}\,\mathcal{E}^{\circlearrowleft}(x_\alpha)\,, \quad (3.73)$$

where Π_n is the set of all partitions α of the set $\{1,\ldots,n\}$ into disjoint union of subsets. For any partition $\alpha\in\Pi_n$ with $\ell=\ell(\alpha)$ blocks, x_α is the vector of size ℓ formed by sums of x_i over the blocks of α.

For $n=1$ formula (3.73) is very simple: $\eta^{|1)}(x) = \mathcal{E}^{\circlearrowleft}(x) = \mathcal{E}(x) = \frac{1}{2\sqrt{\pi}}\frac{e^{\frac{x^3}{12}}}{x^{3/2}}$ and it is exactly what we obtained in section 3.4.2.

Note that Okounkov used the time variables that differ from those typically used in KdV by the factor of $(2k+1)!!$ for the $(2k+1)$th time variable[7]. In order to reproduce these factors in the definition of the resolvent, one has to use here the Laplace-transformed resolvent $\eta(x)$ instead of $\rho(z)$, since under the Laplace transform

$$x^{k+1/2} \longrightarrow \frac{2^{k+1}}{\sqrt{\pi}(2k+1)!!}\frac{1}{z^{k+3/2}} \quad (3.74)$$

3.4.4 L_{-1} Virasoro constraint and genus-zero resolvents

In this section we show explicitly that Okounkov's functions (i.e. the r.h.s. of (3.73)) satisfy the lowest L_{-1} Virasoro constraint. Note that, when proving (3.34) in sect.3.2.3, we used only the leading asymptotics of the recursive relations in z, which is equivalent to the L_{-1}-constraint. In fact, the L_{-1}-constraint is sufficient to determine the genus zero resolvents in the case when only $T_3\neq 0$. Hence by demonstrating the function satisfies the L_{-1}-constraint one is automatically guaranteed the genus zero result is correct (therefore, it is possible just to apply the same arguments as in proof of (3.34)).

The L_{-1}-constraint imposed on the partition function is equivalent to the following constraint on the asymptotics of the ρ-resolvents (compare with (3.41)):

$$\rho^{|n)}(z_1,\ldots,z_n) = -\frac{2}{z_n^{3/2}}\left(\sum_{i=1}^{n-1}\partial_i\right)\rho^{|n-1)}(z_1,\ldots,z_{n-1}) + o(z_n^{-3/2}),\quad z_n\to\infty \quad (3.75)$$

Under the Laplace transform, (3.59) it leads to

$$\eta^{|n)}(x_1,\cdots,x_n) = \sqrt{\frac{x_n}{\pi}}\,(x_1+\cdots+x_{n-1})\,\eta^{|n-1)}(x_1,..,x_{n-1}) + o(\sqrt{x_n}),\quad x_n\to 0 \quad (3.76)$$

[6]In [16] Okounkov used notation $\mathcal{G}$ instead of η.

[7]We do not care here about some k-independent factor, because it can be easily eliminated by rescaling.

i.e. the η-resolvents given by formula (3.73) satisfy the L_{-1} constraint.

In order to prove this formula, one needs to use the following identity

$$
\begin{aligned}
&\int_0^\infty da \int_0^\infty db\, f(a,b)\, \frac{e^{-\frac{(a-b)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}} = \\
&= \int_0^\infty dc\, f(c,c) - \sqrt{\frac{\epsilon}{2\pi}}\, f(0,0) + O(\epsilon) \equiv \int_0^\infty dc \left\{ f(c,c) + \sqrt{\frac{\epsilon}{2\pi}}\, \partial_c f(c,c) \right\} + O(\epsilon)
\end{aligned}
\tag{3.77}
$$

It follows from the decomposition

$$
f(a,b) = [f(a,b) - \Theta(\Lambda - (a+b)) f(0,0)] + f(0,0)\, \Theta(\Lambda - (a+b)) \tag{3.78}
$$

where Θ is the Heaviside step function. The difference in the brackets can be proved not to contain the $O(\sqrt{\epsilon})$ term, while the remaining integral can be computed exactly.

Now, one can apply to the Laplace transform, (3.59) of the ρ-resolvents (3.75) formula (3.77) with $\epsilon = 2x_k$

$$
\begin{aligned}
&\mathcal{E}(x_1,\ldots,x_n) = \frac{1}{2^n \pi^{n/2}} \frac{\exp\left(\frac{1}{12}\sum x_i^3\right)}{\prod \sqrt{x_i}} \times \\
&\quad \int_0^\infty \prod_{i=1}^n ds_i\, e^{\left\{\cdots - \frac{(s_{k-1}-s_k)^2}{4x_{k-1}} - \frac{(s_k - s_{k+1})^2}{4x_k} - \frac{(s_{k+1}-s_{k+2})^2}{4x_{k+1}} - \cdots - s_k \frac{x_k + x_{k-1}}{2} - s_{k+1}\frac{x_{k+1}+x_k}{2} \cdots \right\}} \underset{x_k \to 0}{=}
\end{aligned}
\tag{3.79}
$$

$$
= \frac{1}{2^{n-1}\pi^{(n-1)/2}} \frac{\exp\left(\frac{1}{12}\sum\limits_{i\neq k} x_i^3\right)}{\prod\limits_{i\neq k}\sqrt{x_i}} \int_0^\infty \prod_{i\neq k,k+1} ds_i\, ds\, A\, e^{\left\{\cdots - \frac{(s_{k-1}-s)^2}{4x_{k-1}} - \frac{(s-s_{k+2})^2}{4x_{k+1}} - \cdots - s\frac{x_{k+1}+x_{k-1}}{2}\cdots\right\}} \tag{3.80}
$$

where

$$
A = 1 + \sqrt{\frac{x_k}{\pi}}\, \partial_s + o(\sqrt{x_k}) = 1 - \sqrt{\frac{x_k}{\pi}} \left[\frac{s - s_{k+2}}{2x_{k+1}} - \frac{s_{k-1} - s}{2x_{k-1}} + \frac{x_{k-1} + x_{k+1}}{2} \right] + o(\sqrt{x_k}) \tag{3.81}
$$

In $\mathcal{E}^{\circlearrowleft}$ some terms cancels

$$
\mathcal{E}^{\circlearrowleft}(x_1,\cdots,x_n) = \left[(n-1) - \sqrt{\frac{x_n}{\pi}} (x_1 + \cdots + x_{n-1}) \right] \mathcal{E}^{\circlearrowleft}(x_1,\cdots,x_{n-1}) + o(\sqrt{x_n}) \tag{3.82}
$$

Note that if α is a partition (see [16]), then there are two possibilities: 1) it does not contain the block $\{x_n\}$, then

$$
\mathcal{E}^{\circlearrowleft}(x_\alpha) = \mathcal{E}^{\circlearrowleft}(x_\alpha|_{x_n=0}) + o(\sqrt{x_n}) \tag{3.83}
$$

2) it contains this block, then

$$
\mathcal{E}^{\circlearrowleft}(x_\alpha) = \left[(\ell(\alpha) - 1) - \sqrt{\frac{x_n}{\pi}} (x_1 + \cdots + x_{n-1}) \right] \mathcal{E}^{\circlearrowleft}(x_{\alpha \setminus \{x_n\}}) + o(\sqrt{x_n}) \tag{3.84}
$$

In η further cancelations take place so that finally one arrives at formula (3.77).

As it was already discussed in the beginning of this subsubsection, one can easily derive using (3.76) that

$$
\eta^{(0|k)}(x_1,\ldots,x_k) = \frac{1}{2\,\pi^{k/2}} \left(\sum_{i=1}^k x_i \right)^{k-3} \prod_{i=1}^k x_i^{1/2} \tag{3.85}
$$

Verification of the L_0-constraint needs much enhanced version of (3.77) and much more involved computations.

4 Simplest DV type solution to Kontsevich model: KdV hierarchy

As we already discussed in sect. 2, there are many solutions to the generic Kontsevich model parameterized by an arbitrary function $F[T]$ that satisfies two constraints (2.24), in variance with the Gaussian case of the previous section, when the solution is unique. However, among all these many solutions there is a special family of the so-called Dijkgraaf-Vafa solutions. They are associated with a Riemann surface which genus is generically equal to $N-1$, the number of non-zero times T_k being equal to $N+1$. These solutions are non-generic, and, being extended to depend on higher times T_k, $k > 2N+1$, are required to be associated with the same Riemann surface and to have a smooth limit upon bringing these excessive times to zero in order [42]. Moreover, one typically considers infinitely many excessive times (still keeping the genus $N-1$ of the curve fixed). Then, $F^{(0)}[T]$ is logarithm of the τ-function of a Whitham hierarchy w.r.t. these infinitely many times T_k [26, 43], while the complete matrix model partition function as a function of T_k corresponds to a dispersionful integrable hierarchy.

In this section we consider the simplest DV solution, that is, the solution associated with a sphere. The partition function $Z_K(t)$ (as well as $Z_K[T] \equiv Z_K(t)|_{\tau=0}$) of this system is nothing but a τ-function of the KdV hierarchy, while its planar limit, $F^{(0)}[T]$ is logarithm of the τ-function of the dispersionless KdV hierarchy (i.e. Whitham hierarchy in the case of spherical Riemann surface). This is exactly the solution that was previously considered as relevant to $2d$ gravity [44, 23, 24, 10, 11].

Throughout this section we denote

$$U = \frac{\partial^2 F}{\partial T_1^2} \tag{4.1}$$

and its dispersionless counterpart

$$u = \frac{\partial^2 F^{(0)}}{\partial T_1^2} \tag{4.2}$$

The Lax operator is

$$\mathbf{L} = g^2\partial^2 + 2U \tag{4.3}$$

The evolution is given by the flows

$$\frac{\partial \mathbf{L}}{\partial T_i} = g[\mathbf{L}_+^{\frac{i}{2}}, \mathbf{L}] \tag{4.4}$$

Here $[\cdot]_+$ denotes the differential part of the pseudo-differential operator (i.e. the "non-negative part" in the formal operator ∂). The appropriate solution of the Virasoro constraints is fixed by the string equation

$$[\mathbf{L}, \mathbf{M}] = 2g \tag{4.5}$$

where

$$\mathbf{M} = \sum_{k=1}^{N} (2k+1)T_{2k+1}\,\mathbf{L}_+^{k-1/2} \tag{4.6}$$

4.1 An example: N = 2 case

Let us assume that $Z[T] = e^{F[T]/g^2}$, which we "found" (up to some arbitrary function) in subsection 2.2 solving the first Virasoro constraints, is, in addition, a τ-function of the KdV (or the 2-reduced KP) hierarchy. The 3rd and the 5th equations of the hierarchy are

$$\begin{aligned} \frac{1}{3}\frac{\partial U}{\partial T_3} &= \frac{\partial}{\partial T_1}\left(\frac{U^2}{2} + g^2\frac{U''}{12}\right) \\ \frac{1}{15}\frac{\partial U}{\partial T_5} &= \frac{\partial}{\partial T_1}\left(\frac{U^3}{6} + g^2\frac{UU''}{12} + g^2\frac{U'^2}{24} + g^4\frac{U''''}{240}\right) \end{aligned} \tag{4.7}$$

Picking up F from (2.37) and inserting it into these equations leads to the Painlevé-I equation on $G(\eta_2) \equiv \tilde{\tilde{F}}''(\eta_2)$:

$$g^2 G'' = -6 \cdot G^2 + \frac{3^5 5^4}{4}\eta_2 \tag{4.8}$$

Solving it perturbatively w.r.t. g^2, one obtains series (2.40) with the coefficients being recursively determined[8]. The first two coefficients coincide with those in (2.38). From that equation it follows that $\tilde{\tilde{F}}$ may also have an extra term $C'\eta_2$. However, one can check that actually $C' = 0$ using e.g. the equation

$$\mathrm{res}\mathbf{L}^{5/2} = \frac{\partial^2 F}{\partial T_1 \partial T_5} \tag{4.9}$$

where the residue is defined by

$$\mathrm{res} \sum_{k=-\infty}^{\infty} a_k (g\partial)^k = a_{-1} \tag{4.10}$$

Thus, we have fixed all $F^{(p)}[T]$ completely, there is no more ambiguity and the curve and the densities are fixed. As we mentioned, $F^{(0)}[T]$ is such that the elliptic curve actually degenerates:

$$y^2 = (5T_5)^2 \left(x + \frac{1}{5}\frac{\sqrt{T_3^2 - \frac{10}{3}T_1T_5} + 2T_3}{T_5} \right)^2 \left(x - \frac{2}{5}\frac{\sqrt{T_3^2 - \frac{10}{3}T_1T_5} - T_3}{T_5} \right) \tag{4.11}$$

i.e. the torus is pinched at the point $y = 0,\ x = -\frac{1}{5}\frac{\sqrt{T_3^2 - \frac{10}{3}T_1T_5} + 2T_3}{T_5}$. Thus the curve is equivalent to the rational one $Y^2 = x - \frac{2}{5}\frac{\sqrt{T_3^2 - \frac{10}{3}T_1T_5} - T_3}{T_5}$.

A few first (multi)-densities are (here we consider the critical point[9] case $T_3 = 0,\ T_5 = \frac{-2}{15}$ to make formulae more compact):

$$y(z) = -\frac{2}{3}(\sqrt{T_1} + z)Y(z), \quad Y^2(z) = z - 2\sqrt{T_1} \tag{4.12}$$

$$\rho^{(0|2)}(z_1, z_2) = -\frac{4\sqrt{T_1} - z_1 - z_2 + \frac{z_1+z_2}{\sqrt{z_1z_2}}Y(z_1)Y(z_2)}{Y(z_1)Y(z_2)(z_1 - z_2)^2} = \rho^{(0|2)}_{(\mathrm{N}=1)}(z_1, z_2)\Big|_{s=2\sqrt{T_1}} \tag{4.13}$$

$$\rho^{(0|3)}(z_1, z_2, z_3) = \frac{1}{2\sqrt{T_1}}\frac{1}{Y^3(z_1)Y^3(z_2)Y^3(z_3)} \tag{4.14}$$

$$\rho^{(0|4)}(z_1, z_2, z_3, z_4) = -\frac{1}{4\,T_1^{3/2}Y^5(z_1)Y^5(z_2)Y^5(z_3)Y^5(z_4)}(z_1z_2z_3z_4 - 5\,T_1^{1/2}(z_1z_2z_3 + z_1z_2z_4 + z_1z_3z_4 + z_2z_3z_4) +$$
$$+ 16\,T_1(z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4) - 44\,T_1^{3/2}(z_1 + z_2 + z_3 + z_4) + 112\,T_1^2) \tag{4.15}$$

$$\rho^{(1|1)}(z) = -\frac{1}{48 \cdot T_1}\frac{z - 5\sqrt{T_1}}{Y^5(z)} \tag{4.16}$$

$$\rho^{(2|1)}(z) = -\frac{7}{9 \cdot 2^{10} \cdot T_1^{7/2}}\frac{613T_1^2 - 503T_1^{3/2}z + 204T_1z^2 - 44T_1^{1/2}z^3 + 4z^4}{Y^{11}(z)} \tag{4.17}$$

---

[8]Note that, strictly speaking, there are two solutions to (4.8) corresponding to the two branches of $\eta_2^{5/2}$ in (2.40) (or, equivalently, to the two branches of $\sqrt{T_3^2 - \frac{10}{3}T_1T_5}$ in the expressions below). But only one of these branches has a smooth limit $T_5 \to 0$, i.e. permits the transition $N = 2 \to N = 1$ considered in subsection 2.3. Therefore, only one of them is associated with the DV solution.

[9]By critical point we mean the case when $T_{2N+1} = \mathrm{const}$ and other $T_i = 0,\ i > 1$. T_1 is as usual a variable.

$$\rho^{(1|2)}(z_1,z_2) = -\frac{1}{3\cdot 2^7\cdot T_1^4\cdot Y^7(z_1)Y^7(z_2)}\cdot$$
$$\cdot\Big(2z_1^2z_2^2 - 14\sqrt{T_1}(z_1^2z_2+z_2^2z_1) + 35T_1(z_1^2+z_2^2) + 89T_1z_1z_2 - 182T_1^{3/2}(z_1+z_2) + 284T_1^2\Big) \tag{4.18}$$

Thus, we checked that if $Z = e^{\mathcal{F}/g^2}$ satisfies the Virasoro constraints (for $N = 2$), then the following statements are equivalent:

- Z is a τ-function of the KdV-hierarchy
- The elliptic curve $y(z)$ degenerates into the rational one and the multi-densities $\rho^{(p|m)}$ have no poles at the two marked points on the sphere which come from the double point singularity on the torus.

Equivalently, one can say that the poles are only at $Y = 0$ points. Thus, the condition imposed on the curve determines $F^{(0)}[T]$ and then the condition of canceling the singularities determines $F^{(p)}[T]$, $p > 0$ (for general $F^{(p)}[T]$ the poles at $x = -\frac{1}{5}\frac{\sqrt{T_3^2-\frac{10}{3}T_1T_5}+2T_3}{T_5}$ do exist). We will specify this statement for general case below.

4.2 Quasiclassical limit of the KdV hierarchy as Whitham hierarchy

In this subsection we review some features of the quasiclassical (or dispersionless) limit of the KdV hierarchy and its representation in terms of the generalized Whitham hierarchy [45, 46]. We work only quasiclassically and suppress the corresponding (0) superscript. In the quasiclassical limit of the KdV hierarchy, the momentum is just a commuting variable: $g\partial \rightsquigarrow P$, because one can neglect all the commutators $[P,\cdot] \sim g$. Then, the Lax operator is just a function

$$L = P^2 + 2u \tag{4.19}$$

which satisfy an additional constraint – the string equation:

$$\{L, M\} = 2 \tag{4.20}$$

where $\{\cdot,\cdot\}$ is the Poisson brackets ($\{P, T_1\} = 1$) and

$$M = \sum_{k=1}^{N}(2k+1)T_{2k+1}\,L_+^{k-1/2} = \sum_{j=0}^{N-1} M_j P^{2j+1} \tag{4.21}$$

By $[\cdot]_+$ we denote the positive part of expansion at the vicinity of $P = \infty$. T_k and $\frac{\partial F}{\partial T_k}$ can be represented as follows (see proof in Appendix A):

$$\begin{aligned} T_k &= -\frac{1}{k}\mathrm{res}_{P=\infty}\left\{P\,ML^{-k/2}dP\right\} \\ \frac{\partial F}{\partial T_k} &= -\mathrm{res}_{P=\infty}\left\{P\,ML^{k/2}dP\right\} \end{aligned} \tag{4.22}$$

This defines our system as generalized Whitham hierarchy for the sphere parameterized by P.

For generic N, the curve (2.23) also degenerates to the rational one, when one imposes on Z an additional constraint to be the τ-function of the KdV hierarchy. This is actually an obvious consequence of the equivalence of the curve (2.23) and the one appearing in the Whitham hierarchy, since the dispersionless limit of KdV (which describes $F^{(0)}[T]$ in (2.23)) corresponds to the Whitham hierarchy on the sphere. Hence, **imposing on the solution of KdV hierarchy additional Virasoro constraints quasiclassically is equivalent to total degeneration of the corresponding curve $\mathcal{C}_{2,2N-1}$ given by (2.23). This degenerated curve has the global parametrization $z = L(P)$, $y = M(P)$.**

Indeed, let us denote

$$x = P^2 + 2u = L; \quad \tilde{y} = \sum_{k=0}^{N-1} M_k P^{2k+1} \tag{4.23}$$

Then

$$\tilde{y}^2(x) = (x - 2u)\left(\sum_{k=0}^{N-1} M_k (x - 2u)^k\right)^2 \tag{4.24}$$

The curve $C_{2,2N-1}$ defined earlier reads explicitly as

$$y^2 = \frac{1}{z}\left\{\left(\sum_{k=0}^{N} z^k (2k+1) T_{2k+1}\right)^2 - T_1^2\right\} + 2\sum_{m=0}^{N-2}\sum_{k=m+2}^{N} z^{k-m-2}(2k+1)T_{2k+1}\frac{\partial F}{\partial t_{2m+1}} \tag{4.25}$$

To show that $y^2(x) = \tilde{y}^2(x)$, one can use formulae (4.22)

$$T_k = -\frac{1}{2k}\mathrm{res}_{P=\infty}\left[x^{-k/2}dS\right] \tag{4.26}$$

$$\frac{\partial F}{\partial T_k} = -\frac{1}{2}\mathrm{res}_{P=\infty}\left[x^{k/2}dS\right] \tag{4.27}$$

where $dS = \tilde{y}dx$. Then, expression (4.25) can be combined into the sum

$$y^2(z) = \frac{1}{4}\sum_{a+b>0} z^{a+b-1}\mathrm{res}_{P=\infty}\left[x^{-a-1/2}dS\right]\mathrm{res}_{P=\infty}\left[x^{-b-1/2}dS\right] = [y_R^2(z)]_+ \tag{4.28}$$

where

$$y_R(z) = -\frac{1}{2}\sum_a z^{a-1/2}\mathrm{res}_{P=\infty}\left[x^{-a-1/2}dS\right] = -\sum_a z^{a-1/2}\mathrm{res}_{x=\infty}\left[x^{1/2}\tilde{y}(x)\frac{dx}{x^{a+1}}\right] = \tilde{y}(z) \tag{4.29}$$

Therefore

$$y^2(z) = [\tilde{y}^2(z)]_+ = \tilde{y}^2(z) \tag{4.30}$$

The inverse is also true. The condition of curve degeneration into sphere gives us $N-1$ independent linear PDEs on F (pinching of all $N-1$ handles, or coinciding of the corresponding $N-1$ pairs of roots) which fix completely the ambiguity, the function $\widetilde{F}^{(0)}$ of $N-1$ variables.

When all T_k's are fixed, one can completely determine all M_k's and u. That is, u is defined by the equation

$$\sum_{k=0}^{N}\frac{(2k+1)!!}{k!}T_{2k+1}\,u^k = 0 \tag{4.31}$$

which has only N solutions. Thus, in generic case one obtains[10]

$$y(x) = P(x)y_r(x) \quad \left(P(x) = \sqrt{x-2u}, \quad y_r(x) = \sum_{s=0}^{N-1} M_s(x-2u)^s\right) \tag{4.32}$$

[10] $Y(x)$ of subsection 4.1 is just $P(x)$.

4.3 An alternative way to obtain multi-resolvents

When $T_1, \ldots, T_{2N+1}$ times are turned on, there are N solutions to the equations of the KdV hierarchy that satisfy the Virasoro constraints. The choice of solution corresponds to the choice of a root in equation (4.31). $N-1$ of these solutions have a smooth limit when $T_{2N+1} \to 0$, and one of them (corresponding to the root of (4.31) which goes to infinity) diverges. However, only one solution survives when all the times $T_k \to 0$, $k > 3$ reducing in this limit to the Gaussian model. Putting this differently, one may associate T_k's with τ_k's of the Gaussian model in this case. Partition functions for different choices of T_k's are actually given by the same function in the sense that

$$Z(T', \tau') = Z(T, \tau) = Z_K(t) \quad if \quad T'_{2k+1} + \tau'_{2k+1} = T_{2k+1} + \tau_{2k+1} = t_{2k+1} \,. \tag{4.33}$$

However, different solutions, i.e. different functions $F[T_1, \ldots, T_{2N+1}]$, can be obtained from this one by analytic continuation with respect to variables[11] $T_1, \ldots, T_{2N+1}$.

In this subsection we explain how to derive formulae analogous to (3.34) and (3.36) for multi-resolvents for the general DV solution considered in this section. This derivation is closer to the original way of getting (3.34) in [20] (see also [47]). The crucial point for the derivation is to use specific *moment* variables [14]. The partition function in these variables can be easily obtained within the realization of the Kontsevich partition function as the highest weight of the Virasoro algebra given on the spectral curve (3.6), [6]. Within this approach, one makes a change of the local parameter on the spectral curve[12] $z \to G^{2/3}(z - 2\mathfrak{u})$ (where G and $\mathfrak{u}$ are some functions of times to be fixed yet) [13] that generates the change of times to the moment variables,

$$\tilde{t}_{2m+1} = \frac{1}{(2m+1)G^{\frac{2m+1}{3}}} \oint \frac{(v(z) + W(z))dz}{(z - 2\mathfrak{u})^{m+\frac{1}{2}}} \tag{4.34}$$

Then, the partition functions in old and new variables are related by the formula (see [6, sect.3] for the detailed definitions and derivations):

$$Z_K(t) = G^{-\frac{1}{24}} e^{U_{KK}(t)} Z_K(\tilde{t}) \tag{4.35}$$

This formula is correct for any functions G and $\mathfrak{u}$. We specify them so that

$$\tilde{\tau}_1 = \tilde{\tau}_3 = 0, \quad (\tilde{\tau}_{2k+1} - \frac{1}{3}\delta_{k,1} = \tilde{t}_{2k+1}) \tag{4.36}$$

Then,

$$\begin{gathered} \mathcal{F}^{(0)} = U_{KK}(t) = \frac{1}{2} \oint_\infty \oint_\infty \rho^{(0|2)}(z_1, z_2|2\mathfrak{u})(\phi(z_1) + \Phi(z_1))(\phi(z_2) + \Phi(z_2))dz_1 dz_2 \\ \mathcal{F}^{(1)} = -\frac{1}{24}\log(G) \end{gathered} \tag{4.37}$$

where $\rho^{(0|2)}(z_1, z_2|2\mathfrak{u})$ is given by formula (3.18) and $\phi(z) = \sum \tau_k z^{k+1/2}$, $\Phi(z) = \sum T_k z^{k+1/2}$, $2\phi'(z) = v(z)$, $2\Phi'(z) = W(z)$

Introduce now, after [20], the function

$$S(a) := \oint \frac{(v(z) + W(z))dz}{\sqrt{z - 2a}} = \sum_{k=0}^{\infty} \frac{(2k+1)!!}{k!} t_{2k+1}\, a^k \tag{4.38}$$

[11] We use it implicitly in what follows. E.g., formulae (4.35)-(4.37) are known for the Gaussian branch. We derive from them explicit formulas for $\rho^{(p|k)}$'s, e.g., (4.52). They are rational functions in u and M_k's (which in turn are polynomials in u and linear functions in T_k's). Once one have obtained these formulae for the specific solution to the KdV equations, one can make analytic continuation in $T_1, \ldots, T_{2N+1}$ to prove that they hold on over the branches.

[12] Note that the spectral curve considered in [6] is double covering of curve (3.6) considered here. Hence, the local parameter ξ of [6] is related to the parameter z here as $\xi^2 = z$.

[13] As it will be seen later $u = \mathfrak{u}|_{\tau=0}$ coincides with u appeared earlier in section 4.

Then, from (4.34) it follows that

$$S(\mathfrak{u}) = 0$$
$$S'(\mathfrak{u}) := \frac{\partial S(a)}{\partial a}\Big|_{a=\mathfrak{u}} = -G \tag{4.39}$$

and, for $k > 1$,

$$\tilde{\tau}_{2k+1} = \frac{S^{(k)}}{(2k+1)!!(-S')^{\frac{2k+1}{3}}}, \tag{4.40}$$

where $S^{(k)} = S^{(k)}(u)$. Now one can represent the operator $\nabla(z)$ in terms of the derivative with respect to $\mathfrak{u}$

$$\nabla(x) = f(x)\Omega + \frac{\partial}{\partial \tau}(x) \tag{4.41}$$

$$f(x) = \frac{1}{(x-2\mathfrak{u})^{\frac{3}{2}}}$$
$$\Omega = -\frac{1}{S'}\frac{d}{d\mathfrak{u}} \tag{4.42}$$

where operator $\frac{\partial}{\partial \tau}(x)$ acts only on explicit dependence of $S^{(k)}$ on τ's. Then,

$$\nabla(x)\mathcal{F}^{(0)} = \oint \rho^{(0|2)}(x,z|2\mathfrak{u})(\phi(z)+\Phi(z))dz - \frac{1}{S'}\frac{1}{(x-2\mathfrak{u})^{\frac{3}{2}}}\frac{\partial}{\partial \mathfrak{u}}\mathcal{F}^{(0)} \tag{4.43}$$

The derivative of the two-point function splits

$$\frac{\partial}{\partial \mathfrak{u}}\rho^{(0|2)}(z_1,z_2|2\mathfrak{u}) = \frac{1}{(z_1-2\mathfrak{u})^{\frac{3}{2}}(z_2-2\mathfrak{u})^{\frac{3}{2}}} \tag{4.44}$$

and

$$\frac{\partial}{\partial \mathfrak{u}}\mathcal{F}^{(0)} \sim \left(\oint \frac{(\phi(x)+\Phi(x))dx}{(x-2\mathfrak{u})^{\frac{3}{2}}}\right)^2 \sim (\tilde{\tau}_1)^2 = 0 \tag{4.45}$$

In this way one gets the well-known result, which is general for matrix models: the second derivative of the planar free energy depends only on ramification points (parameter $\mathfrak{u}$ in our case)

$$\nabla(z_1)\nabla(z_2)\mathcal{F}^{(0)} = \rho^{(0|2)}(z_1,z_2|2\mathfrak{u}) \tag{4.46}$$

Acting again with the operator $\nabla(z_3)$, one gets

$$\nabla(z_1)\nabla(z_2)\nabla(z_3)\mathcal{F}^{(0)} = -\frac{1}{S'}\frac{1}{((z_1-2\mathfrak{u})(z_2-2\mathfrak{u})(z_3-2\mathfrak{u}))^{\frac{3}{2}}} \tag{4.47}$$

From the observation that $[\Omega, \nabla(x)] = 0$ with help of the relation

$$\nabla(z_1)\cdots\nabla(z_k)(S')^{-1} = \Omega^k (S')^{-1}\prod_{i=1}^{k} f(z_k) \tag{4.48}$$

one derives

$$\nabla(z_1)\cdots\nabla(z_k)\mathcal{F}^{(0)} = -\left(-\frac{1}{S'}\frac{\partial}{\partial \mathfrak{u}}\right)^{k-3}\frac{1}{S'}\prod_{i=1}^{k} f(z_i) \tag{4.49}$$

where, if one wants to get the expression for zero times τ, one should substitute

$$S^{(s)}|_{\tau=0} = \sum_{k=s}^{N}\frac{(2k+1)!!}{(k-s)!}T_{2k+1}u^k = M_{s-1} \tag{4.50}$$

$$\mathfrak{u}|_{\tau=0} = u, \quad \sum_{k=0}^{N} \frac{(2k+1)!!}{k!} T_{2k+1}\, u^k = 0 \tag{4.51}$$

which coincides with the string equation, (4.31). Thus, we obtain the following generalization of (3.34):

$$\rho^{(0|k)}(z_1, \cdots, z_k) = -\left(-\frac{1}{M_0}\frac{\partial}{\partial u}\right)^{k-3} \frac{1}{M_0} \prod_{i=1}^{k} \frac{1}{(z_i - 2u)^{3/2}} \tag{4.52}$$

Let us note that $\frac{\partial M_s}{\partial u} = (2s+3)M_{s+1}$ and M_s, u have a nice description in terms of the curve (see e.g. (4.32)). From (4.37), (4.39) and (4.47) one gets

$$\mathcal{F}^{(1)} = -\frac{1}{24}\log S' = -\frac{1}{24}\log\left(\frac{\partial^3}{\partial\tau_0^3}\mathcal{F}^{(0)}\right) \tag{4.53}$$

For higher genera it is probably too naive to expect expressions for multi-resolvents to be same simple as for genera 0 and 1, but one can get (less simple) explicit expressions. If the variables u and $J_k := S^{(k)}$ for $k > 0$ are considered as independent, then

$$\begin{aligned} \frac{\partial}{\partial\tau}(x) &= \sum_{k=1} f^{(k)}(x)\frac{\partial}{\partial J_k} \\ \frac{d}{d\mathfrak{u}} &= \frac{\partial}{\partial\mathfrak{u}} + \sum_{k=1} J_{k+1}\frac{\partial}{\partial J_k} \end{aligned} \tag{4.54}$$

with $f^{(k)}(x) = \frac{\partial^k}{\partial\mathfrak{u}^k} f(x)$. Thus

$$\nabla(x) = -\frac{f(x)}{J_1}\frac{\partial}{\partial\mathfrak{u}} + \sum_{k=1}\left(f^{(k)}(x) - \frac{J_{k+1}f(x)}{J_1}\right)\frac{\partial}{\partial J_k} \tag{4.55}$$

This operator is rather easy to apply to low genera, because in terms of the variables J_k, expressions for free energies are simple, for instance,

$$\mathcal{F}^{(2)} = \frac{1}{9216}\left(-\frac{J_4}{J_1^3} + \frac{29J_2J_3}{J_1^4} - \frac{7J_2^3}{5J_1^5}\right) \tag{4.56}$$

$$\begin{aligned} \nabla(x)\mathcal{F}^{(2)} = \frac{1}{2}\left(f^{(1)}(x) - \frac{J_2 f(x)}{J_1}\right)\left(\frac{1}{192}\frac{J_4}{{J_1}^4} - \frac{29}{720}\frac{J_2J_3}{{J_1}^5} + \frac{7}{144}\frac{{J_2}^3}{{J_1}^6}\right) + \\ +\frac{1}{2}\left(f^{(2)}(x) - \frac{J_3 f(x)}{J_1}\right)\left(\frac{29}{2880}\frac{J_3}{{J_1}^4} - \frac{7}{240}\frac{{J_2}^2}{{J_1}^5}\right) + \frac{29J_2}{5760J_1^4}\left(f^{(3)}(x) - \frac{J_4 f(x)}{J_1}\right) - \\ -\frac{1}{1152J_1^3}\left(f^{(4)}(x) - \frac{J_5 f(x)}{J_1}\right) \end{aligned} \tag{4.57}$$

For the Gaussian branch $M_k = M\delta_{0,k}$, $2u = s$, this gives

$$\rho^{(2|1)}(x) = \nabla(x)\mathcal{F}^{(2)}\Big|_{\tau=0} = \frac{f^{(4)}(x)}{1152\,M^3} = \frac{105}{128\,M^3\,(x-s)^{\frac{11}{2}}} \tag{4.58}$$

One can easily see that, in the generic DV case, $\rho^{(p|m)}(x_1, \ldots, x_m)$ can be expressed through the coefficients $M_0, \ldots, M_{N-1}$ and $P(x)$ only. From formula (4.55) it follows that the singularities in x_i can only be of the form $f^{(k)}(x_i)|_{\tau=0}$, i.e. $1/P^{2k+3}(x_i)$.

Here we list several first densities (for general N) (using variables $u, M_0, \ldots, M_{N-1}$ instead of $T_1, \ldots, T_{2N+1}$):

$$\rho^{(0|1)}(x) = -W(x)/\sqrt{x} - P(x)\cdot y_r(x) \tag{4.59}$$

$$\rho^{(0|2)}(x_1,x_2) = \frac{1}{(x_1-x_2)^2}\left\{\frac{P^2(x_1)+P^2(x_2)}{P(x_1)P(x_2)} - \frac{x_1+x_2}{\sqrt{x_1x_2}}\right\} \tag{4.60}$$

$$\rho^{(0|3)}(x_1,x_2,x_3) = \frac{-1}{M_0}\frac{1}{P^3(x_1)P^3(x_2)P^3(x_3)} \tag{4.61}$$

$$\begin{aligned}\rho^{(0|4)}(x_1,x_2,x_3,x_4) = \frac{3}{M_0^2}\frac{1}{P^3(x_1)P^3(x_2)P^3(x_3)P^3(x_4)}\cdot \\ \cdot\left(\frac{1}{P^2(x_1)}+\frac{1}{P^2(x_2)}+\frac{1}{P^2(x_3)}+\frac{1}{P^2(x_4)}-\frac{M_1}{M_0}\right)\end{aligned} \tag{4.62}$$

$$\rho^{(1|1)}(x) = \frac{-1}{8M_0^2}\frac{M_0-M_1P^2(x)}{P^5(x)} \tag{4.63}$$

Note that the dependence on x and u enters only through the difference $x-2u = P^2(x)$ (analogously to what we had in the $N=1$ case (see formula (3.8)), again except for the non-meromorphic parts in $\rho^{(0|1)}$ and $\rho^{(0|2)}$). This type of solutions to the Virasoro constraints is sometimes referred to as one-cut solutions (for the obvious reason).

4.4 KdV/DV solution and absence of poles

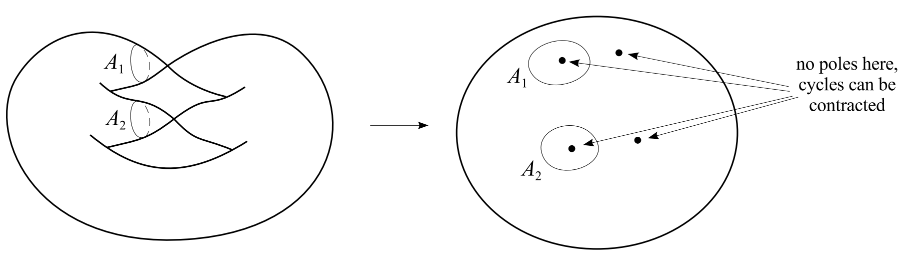
Taking into account what was said at the end of the previous subsection, one can state that **if $Z(T)$ satisfies the reduced Virasoro constraints $\check{L}_{-1},\ \check{L}_0$, then $Z[T]=\tau_{\mathbf{KP}} \iff$ the curve $\mathcal{C}_{2,2N-1}$ degenerates to a sphere and $\rho^{(p|m)}$ for all p,m have no poles at marked points on the sphere which came from degenerated handles, i.e. at the zeroes of $y_r(x)$.**

Proof of the genus-zero (regarding degeneration of the curve) part of this conjecture is presented in s.4.2. It is actually enough to check the condition for higher p only for the one-point functions $\rho^{(p|1)}(x)$, because when performing iterations (2.20) the singularities just can not arise in the multi-densities $\rho^{(p|m)}(x_1,..,x_m)$ due to the symmetry between $x_1,..,x_m$. Then, for each p we have the following condition: the numerator of $\rho^{(p|1)}(x)$ obtained through (2.20), which is a polynomial in x, should be divisible by $y_r(x)$ in the denominator. In the general case, the remainder is a polynomial of degree $N-2$, and thus one has $N-1$ equations, which are the first order linear PDEs on $F^{(p)}$. Hence, the function $\widetilde{F}$ of $N-1$ variables is completely determined.

The absence of the poles in all $\rho^{(p|1)}$ is equivalent to vanishing of the integrals

$$\oint_{A_i}\rho^{|1)}(x)dx = 0, \qquad\qquad \forall i = 1\ldots N-1 \tag{4.64}$$

where the A-cycles encircle handles (which are actually pinched):



and $\rho^{|1)} = \sum_{p=0}^{\infty} g^{2p}\rho^{(p|1)}$ is the total one-point function. Therefore, one can actually forget about this points and the initial curve, and work in terms of the reduced curve $P(x)=\sqrt{x-2u}$.

4.5 A particular case: Chebyshev spectral curves

There is a special choice of values of T_k which is usually referred to as conformal background [48]. This choice leads to the Chebyshev curves[14]. Indeed, **if values of times $T_k = T_k(\lambda)$ are given by**

$$\sum_{k=0}^{N} T_{2k+1}\xi^{k+1/2}\frac{(2k+1)!!}{2^{2k-1}} = \frac{2N-1}{\sqrt{2\pi}}(-\lambda)^{N-1/2}\left[K_{N-1/2}(-\lambda/\xi)\xi\right]_+ \tag{4.65}$$

then there exists a solution $u = -\frac{\lambda}{4}$ of the string equation (4.31) such that

$$x = P^2 - \frac{1}{2}\lambda = \frac{\lambda}{2}\,\mathrm{T}_2(P/\sqrt{\lambda}) \tag{4.66}$$

$$y = \lambda^{N-1/2}\,\mathrm{T}_{2N-1}(P/\sqrt{\lambda}) \tag{4.67}$$

or, equivalently

$$y = \lambda^{N-1/2}\,\mathrm{T}_{N-1/2}\left(\frac{2x}{\lambda}\right) \tag{4.68}$$

We denote here by T_* the Chebyshev polynomials. The choice of times (4.65) defines the conformal background.

In order to prove this claim, one can set $\lambda = 1$ without loss of generality. According to what is said at the end of s.4.2, the space of curves of type (4.23)-(4.24) covers N times the space of T_k's. Then, it is enough to prove that the times given by the Chebyshev curve $y = \mathrm{T}_{N-1/2}(2x)$ are such that (4.65) satisfies. Taking into account that $T_{2k+1} = -\frac{1}{2k+1}\mathrm{res}_{x=\infty}\left[x^{-k-1/2}y(x)dx\right]$, it is correct if

$$\sum_{k=-\infty}^{N} (\mathrm{T}_\nu)_k\,\Gamma(k+1/2)\,x^{k-1/2} = (-1)^\nu \nu\,K_\nu(-1/x) \qquad (\nu \equiv N-1/2) \tag{4.69}$$

is correct. We denote here by K_* the modified Bessel functions of the second kind and $\mathrm{T}_\nu(x) \equiv \sum_k (\mathrm{T}_\nu)_k x^{k-1/2}$. Using the integral representation for the Γ-function and deforming the contour, one can rewrite this equation as

$$\int_{-1}^{+\infty} \mathrm{T}_\nu(\tau)e^{-\tau l}l\,d\tau = (-1)^\nu \nu\,K_\nu(-l) \qquad (l \equiv 1/x > 0) \tag{4.70}$$

Then, integrating by parts in the l.h.s. and using the integral representation for the Bessel function in the r.h.s., one comes to

$$\int_{-1}^{+\infty} \mathrm{T}'_\nu(\tau)e^{-\tau l}\,d\tau = \frac{1}{2}\int_C e^{-\frac{l}{2}(t+\frac{1}{t})}\,\nu t^{\nu-1}dt \tag{4.71}$$

where contour C goes from $0+$ to $+\infty$ encircling 0 one time. One can check this equality just by changing the variables $t + 1/t = \tau$, $t = t_\pm = \tau \pm \sqrt{\tau^2-1}$ and using $\mathrm{T}_\nu(\tau) = (t_+^\nu + t_-^\nu)/2$.

Note that in the general case if one has a curve

$$\mathrm{T}_p(x) = \mathrm{T}_q(y) \tag{4.72}$$

then due to the composition property of the Chebyshev polynomials, $\mathrm{T}_p \circ \mathrm{T}_q = \mathrm{T}_{pq}$ the curve can be parameterized in the obvious way

$$x = \mathrm{T}_q(P) \qquad\qquad y = \mathrm{T}_p(P) \tag{4.73}$$

[14] In [49] it was shown how the Chebyshev curves emerge in Liouville theory.

5 Generalized Kontsevich Model and duality

In this section we review some basic aspects of GKM. The next section contains some more detailed analysis of the $p = 3$ case and includes finding the curve, the first densities and action of the $p - q$ duality on them.

5.1 Generalized Kontsevich Model

Partition function of GKM is given by the following matrix integral ([10]):

$$Z(t) = \frac{1}{\mathcal{N}(A)} \int dX e^{\frac{1}{g^2}\mathrm{Tr}\left\{-\frac{X^{p+1}}{p+1}+A^p X\right\}} \tag{5.1}$$

It is a function of the Miwa variables

$$T_k + \tau_k \equiv t_k = \frac{p}{p+1}\delta_{p+1,k} + \frac{1}{k}\mathrm{Tr}A^{-k} \tag{5.2}$$

Actually it depends only on times t_k, $k \neq 0\,(\mathrm{mod}\, p)$. The Ward identities in this case are the $\mathcal{W}$-constraints ([23, 10])

$$\mathcal{W}_n^{(k)} Z = 0,\ k = 2..p,\ n \geqslant -k+1 \tag{5.3}$$

For $k = 2$ these are the Virasoro constraints $\mathcal{W}_n^{(2)} = \mathcal{L}_n$. One can define the densities analogously to the $p = 2$ case

$$\rho^{(p|m)}(z_1, \ldots, z_m) = \nabla(z_1)\cdots\nabla(z_2)\mathcal{F}^{(p)}|_{\tau=0}, \quad \nabla(z) = \sum_{k=0}^{\infty} \frac{1}{z^{k/p+1}}\frac{\partial}{\partial \tau_k} \tag{5.4}$$

In particular,

$$\rho^{(0|1)}(x) = -W(x) + y(x),\ \ W(x) = \sum_{k=1}^{p+q} k x^{\frac{k}{p}-1} T_k \tag{5.5}$$

and one expects $y(x)$ to satisfy an algebraic equation

$$P_{p,q}(y, x) = 0 \tag{5.6}$$

defining the spectral curve. The polynomial $P_{p,q}(y, x)$ is of degree p in y and of degree q in x. The GKM is known to be relevant for describing the (p, q)-model of $2d$ gravity when the first $p + q$ times are turned on ($T_n = 0,\ n > p + q$). The well known solution to (5.3) is given by the τ-function of the p-reduced KP-hierarchy ([44, 23]). The spectral curve in this case degenerates to the rational one. When one further specifies to the conformal background the curve has the form of [49]

$$T_p(y) = T_q(x). \tag{5.7}$$

5.2 p-q duality

The $p - q$ duality is obvious when one formulates theory in terms of the Douglas equation [44] generalizing (4.5)

$$[\mathbf{L}, \mathbf{M}] = 2g \tag{5.8}$$

where $\mathbf{L}$ and $\mathbf{M}$ are differential operators of orders p and q respectively. However, when one formulates the theory in terms of p-reduced KP-hierarchy (where L plays role of the Lax operator) or in terms of the matrix model, the duality becomes implicit. In [17] an explicit change of time variables $T \leftrightarrow \bar{T}$ of the hierarchy which connects (p, q) and (q, p) models was found. In [19] the $p - q$ duality was considered from the point of view of the GKM.

Within our approach, one first of all expects that the duality relates the functions $\tilde{\tilde{F}}^{(\mathrm{p,q})}$ and $\tilde{\tilde{F}}^{(\mathrm{q,p})}$ (generalizing those considered in s.2.2) which describe ambiguities in the solutions of the $\mathcal{W}$-constrains

(so that fixing them is equivalent to fixing the solution[15]). They both depend on a certain number (equal to the genus of the spectral curve) of combinations of times T, consequently the change of times should map the set of these combinations for the (p,q) model to that of the (q,p) model. Then (after identification of $\tilde{\tilde{F}}^{(\mathrm{p,q})}$ and $\tilde{\tilde{F}}^{(\mathrm{q,p})}$) one expects that the curves for the two models should coincide, that is, there is an isomorphism $\mathcal{C}_{(\mathrm{p,q})} \xrightarrow{\phi} \mathcal{C}_{(\mathrm{q,p})}$:

$$P_{p,q}(y^{(\mathrm{p,q})}, x^{(\mathrm{p,q})}) \sim P_{q,p}(\phi^* y^{(\mathrm{q,p})}, \phi^* x^{(\mathrm{q,p})}) \tag{5.9}$$

In simple cases, this is just the interchange $x \leftrightarrow y$: $\phi^* y^{(\mathrm{q,p})} = x^{(\mathrm{p,q})}$, $\phi^* x^{(\mathrm{q,p})} = y^{(\mathrm{p,q})}$. Note that, as soon as we talk about solutions to the $\mathcal{W}$-constrains only, without further specifying them as KP solutions, in variance with [17, 18, 19] we deal with a more general, "off-KP" duality.

6 Generalized Kontsevich matrix model ($\mathbf{p = 3}$)

6.1 Some general formulae and definitions

Most of what comes up below is quite analogous to the $p = 2$ case, hence, we present it very briefly (one can find some parallel consideration in [50]).

$$\mathcal{W}_n^{(2)} \equiv \mathcal{L}_n = \frac{1}{3}\left(\frac{1}{2g^2}\sum_{i+j=-3n} ijt_it_j + \sum_{i-j=-3n} it_i\frac{\partial}{\partial t_j} + \frac{g^2}{2}\sum_{i+j=3n}\frac{\partial^2}{\partial t_i \partial t_j} + \frac{1}{3}\delta_{n,0}\right) \tag{6.1}$$

$$\mathcal{W}_n^{(3)} = \frac{1}{9g}\left(\frac{1}{3g^2}\sum_{p+q+r=-3n} pqrt_pt_qt_r + \sum_{p+q-r=-3n} pqt_pt_q\frac{\partial}{\partial t_r}\right.$$
$$\left. + g^2\sum_{p-q-r=-3n} pt_p\frac{\partial^2}{\partial t_q\partial t_r} + \frac{g^4}{3}\sum_{p+q+r=3n}\frac{\partial^3}{\partial t_p \partial t_q \partial t_r}\right)$$

$$\mathcal{L}_n Z = 0, \quad n \geqslant -1 \tag{6.2}$$

$$\mathcal{W}_n^{(3)} Z = 0, \quad n \geqslant -2 \tag{6.3}$$

$$t_k = 0, \quad k = 0 \ (\mathrm{mod}\ 3) \tag{6.4}$$

The shift of times: $t_k = \tau_k + T_k$, $T_k \neq 0$ for $k = 1 \dots (q+3)$. Thus, it can be referred to as (3,q) model.

$$\nabla_i(x) := \sum_{k=0}^{\infty}\frac{1}{x^{k+1}}\frac{\partial}{\partial \tau_{3k+i}} \qquad i = 1,2 \tag{6.5}$$

$$v_i(x) = \sum_{k=0}^{\infty}(3k+i)x^k\tau_{3k+i} \qquad W_i(x) = \sum_{k=0}^{1+\left[\frac{q-i}{3}\right]}(3k+i)x^kT_{3k+i} \quad i = 1,2 \tag{6.6}$$

$$\rho_i(x) := \nabla_i(x)\mathcal{F} \quad i = 1,2 \tag{6.7}$$

The loop equations are

$$2g^1\sum_{n=-1}^{\infty}\frac{1}{x^{n+1}}\mathcal{W}_n^{(2)} Z = 2(\tau_1+T_1)(\tau_2+T_2) + \frac{g^2}{3x} + \rho_1(x)\rho_2(x) + \frac{g^2}{2}(\nabla_1(x)\rho_2(x) + \nabla_2(x)\rho_1(x)) +$$
$$+ P_x^-\left\{(u_1(x)+W_1(x))\rho_1(x) + (u_2(x)+W_2(x))\rho_2(x)\right\} = 0$$

[15]Still, fixing the solution is equivalent to fixing periods of all the multi-densities.

$$9g^3\sum_{n=-2}^{\infty}\frac{1}{x^{n+2}}\mathcal{W}_n^{(3)}=\frac{g^6}{3}\nabla_1^3+\frac{g^6}{3x}\nabla_2^3+g^4P_x^-\left\{\frac{(u_1+W_1)}{x}\nabla_2^2+(u_2+W_2)\nabla_1^2\right\}+$$

$$+g^2P_x^-\left\{\frac{(u_1+W_1)^2}{x}\nabla_2+(u_2+W_2)^2\nabla_1\right\}+\frac{8}{3}(\tau_2+T_2)^3+4(\tau_1+T_1)^2(\tau_4+T_4)+\frac{(\tau_1+T_1)^3}{3x}$$

$$=\frac{1}{3}P_{\{x^{-n}|n\in\mathbb{Z}_+\}}\left\{x\,:\left(g^2\nabla-(u+W)\right)^3:\right\}$$

$$\check{\nabla}_i(x)=\sum_{k=0}^{N}\frac{1}{x^{k+1}}\frac{\partial}{\partial T_{3k+i}} \tag{6.8}$$

$$\nabla(x)=\nabla_1(x)x^{-1/3}+\nabla_2(x)x^{-2/3}\quad \rho(x)=\rho_1(x)x^{-1/3}+\rho_2(x)x^{-2/3}=\nabla(x)\mathcal{F} \tag{6.9}$$

$$W(x)=W_1(x)x^{-2/3}+W_2(x)x^{-1/3} \tag{6.10}$$

$$F[T_1,\ldots,T_{3+q}]=\mathcal{F}|_{\tau=0} \tag{6.11}$$

$$\rho^{(p|m)}(z_1,\ldots,z_m)=\nabla(z_1)\cdots\nabla(z_2)\mathcal{F}^{(p)}|_{\tau=0} \tag{6.12}$$

6.2 Some first densities

Let

$$\rho_1^{(0|1)}=-W_2+y_2x^{1/3}\qquad \rho_2^{(0|1)}=-W_1+y_1x^{2/3} \tag{6.13}$$

$$\rho^{(0|1)}(x)=-W(x)+y(x) \tag{6.14}$$

where

$$y_1+y_2=y \tag{6.15}$$

From the definition of $\rho^{(0|1)}$ one has

$$T_{3k+i}=-\frac{1}{3k+i}\mathrm{res}_{x=\infty}\left\{x^{-k-i/3}y(x)dx\right\}\quad,\quad \left.\frac{\partial\mathcal{F}^{(0)}}{\partial\tau_{3k+i}}\right|_{\tau=0}=-\mathrm{res}_{x=\infty}\left\{x^{k+i/3}y(x)dx\right\} \tag{6.16}$$

Substituting[16] (6.13) into the loop equations for genus zero one obtain the following equation for y defining the $\mathcal{C}_{(\mathrm{p}=3,\mathrm{q})}$ curve:

$$y^3+3A(x)y+B(x)=0 \tag{6.17}$$

where A and B are the following *polynomials*:

$$A(x)=-P_x^+\left\{W_1(x)W_2(x)+\left[W_1(x)(\check{\nabla}_1(x)F^{(0)})+W_2(x)(\check{\nabla}_2(x)F^{(0)})\right]\right\}/x \tag{6.18}$$

$$\begin{aligned}B(x)=-P_x^+\Big\{&W_1^3/x+W_2^3+3\left[W_1(\check{\nabla}_2F^{(0)})^2/x+W_2(\check{\nabla}_1F^{(0)})^2\right]+\\&+3\left[W_1^2(\check{\nabla}_2F^{(0)})/x+W_2^2(\check{\nabla}_1F^{(0)})\right]\Big\}/x\end{aligned} \tag{6.19}$$

[16]What follows is actually true only for $q<7$.

There is also the following identity:

$$y_1 y_2 = -A \tag{6.20}$$

which, together with $y_1 + y_2 = y$, allows one to express y_i through y and A.

Some formulae which are useful for computing the higher densities are

$$\nabla_i(x) P_z^- \{u_i(z)h(z)\} = -3x^{\frac{i}{3}} \partial_x x^{-\frac{i}{3}} \left\{ \frac{zP_x^- h(x) - xP_z^- h(z)}{z-x} \right\} + P_z^- \{u_i(z)\nabla_i(x)h(z)\} \tag{6.21}$$

$$y' = -\frac{1}{3}\frac{B' + 3A'y}{y^2 + A} \tag{6.22}$$

An explicit expression for $\rho^{(0|2)}$ in a certain special case can be found in s.6.6.

6.3 Solving reduced constraints and (3,2)↔(2,3) duality

Now consider reduced Virasoro constraints for $q = 2$ coming from $\mathcal{W}^{(3)}_{-2}$, $\mathcal{L}_{-1}$, $\mathcal{L}_0$-constraints accordingly:

$$\frac{4}{3}T_2^3 + 2T_1^2 T_4 + 10 T_2 T_5 \frac{\partial F}{\partial T_1} + 8T_4^2 \frac{\partial F}{\partial T_2} + \frac{5^2}{2} T_5^2 \frac{\partial F}{\partial T_4} = 0 \tag{6.23}$$

$$2T_1 T_2 + 4T_4 \frac{\partial F}{\partial T_1} + 5T_5 \frac{\partial F}{\partial T_2} = 0 \tag{6.24}$$

$$\frac{g^2}{3} + T_1 \frac{\partial F}{\partial T_1} + 2T_2 \frac{\partial F}{\partial T_2} + 4T_4 \frac{\partial F}{\partial T_4} + 5T_5 \frac{\partial F}{\partial T_5} = 0 \tag{6.25}$$

The general solution is as follows

$$F[T_1, T_2, T_4, T_5] = -\frac{g^2}{15}\log T_5 - \frac{1}{5}\frac{T_1\, T_2{}^2}{T_5} + \frac{4}{75}\frac{T_2{}^3\, T_4}{T_5{}^2} - \frac{2}{25}\frac{T_4{}^2\, T_1{}^2}{T_5{}^2} + \frac{16}{125}\frac{T_4{}^3\, T_1\, T_2}{T_5{}^3} - \frac{32}{625}\frac{T_4{}^4\, T_2{}^2}{T_5{}^4}$$
$$- \frac{128}{9375}\frac{T_4{}^6\, T_1}{T_5{}^5} + \frac{512}{46875}\frac{T_4{}^7\, T_2}{T_5{}^6} - \frac{4096}{5859375}\frac{T_4{}^{10}}{T_5{}^8} + \widetilde{F}\left(\frac{125\, T_1\, T_5{}^3 - 100\, T_2\, T_4\, T_5{}^2 + 16\, T_4{}^4}{5^4 \cdot 3^{4/5} \cdot 2^{-5/5} \cdot (-T_5)^{(16/5)}} \right)$$

where $\widetilde{F}$ is an arbitrary function. This is the (3,2) model. The (2,3) model corresponds to the $N^{(p\,=\,2)} = 2$ case considered in subsections 2.2 and 2.3. Let us denote $\bar{T}_1 = T_1^{(\mathrm{N=2})}, \bar{T}_3 = T_3^{(\mathrm{N=2})}, \bar{T}_5 = T_5^{(\mathrm{N=2})}$.

Using the transformations[17] (the same as in [17]):

$$\bar{T}_1 = T_1 - \frac{4}{5}\frac{T_2\, T_4}{T_5} - \frac{9}{20}\frac{T_3{}^2}{T_5} + \frac{18}{25}\frac{T_3\, T_4{}^2}{T_5{}^2} - \frac{4}{25}\frac{T_4{}^4}{T_5{}^3} \tag{6.26}$$

$$\bar{T}_3 = T_3 - \frac{4}{5}\frac{T_4{}^2}{T_5} \tag{6.27}$$

$$\bar{T}_5 = -\frac{2}{3}T_5 \tag{6.28}$$

one can see that

$$\eta_0 \sim \frac{\bar{T}_1 \bar{T}_5 - \frac{3}{10}\bar{T}_3^2}{\bar{T}_5^{6/5}} \sim \frac{125\, T_1\, T_5{}^3 - 100\, T_2\, T_4\, T_5{}^2 + 16\, T_4{}^4}{T_5{}^{(16/5)}} \tag{6.29}$$

[17] The variable T_3 can be considered as auxiliary. All (3,2) quantities are actually independent on it. For instance, one can choose $T_3 = \frac{4}{15}\frac{T_4^2}{T_5}$ so that the last two terms in the r.h.s. of (6.30) cancel each other.

and, thus, the arbitrary functions in the two models can be identified: $\widetilde{F}_{(2,3)} = \widetilde{F}_{(3,2)}$. Moreover, one gets the same relation as in [17]:

$$u_{(3,2)}[T_1, T_2, T_4, T_5] = u_{(2,3)}[\bar{T}_1, \bar{T}_3, \bar{T}_5] + \frac{2}{25}\frac{{T_4}^2}{{T_5}^2} - \frac{3}{10}\frac{T_3}{T_5} \tag{6.30}$$

$$u_{(3,2)} = \frac{\partial^2 F_{(3,2)}}{\partial T_1^2} \qquad\qquad u_{(2,3)} = \frac{\partial^2 F_{(2,3)}}{\partial \bar{T}_1^2} \tag{6.31}$$

Note that we used so far the W constraints only and did not assume that Z is a τ-function of the KP-hierarchy. Therefore, this is an off-KP duality.

6.4 Duality between (2,3) and (3,2) curves

For $N^{(\mathrm{p}=3)} = 1$ (i.e. the $(3,2)$-model) we have the curve

$$y^3 + 3A(x)y + B(x) = 0 \tag{6.32}$$

$$A(x) = -20T_4T_5 \cdot x - (5T_1T_5 + 8T_2T_4) \tag{6.33}$$

$$B(x) = -125T_5^3 \cdot x^2 - (150T_2T_5^2 + 64T_4^3) \cdot x - (60T_2^2T_5 + 48T_1T_4^2 + 75T_5^2\frac{\partial F^{(0)}_{(3,2)}}{\partial T_1}) \tag{6.34}$$

This is an elliptic curve. For $N^{(\mathrm{p}=2)} = 2$ (i.e. the $(2,3)$-model) we had the following elliptic curve:

$$y^2 - 25\bar{T}_5^2 \cdot x^3 - 30\bar{T}_3\bar{T}_5 \cdot x^2 - (10\bar{T}_1\bar{T}_5 + 9\bar{T}_3^2) \cdot x - (6\bar{T}_1\bar{T}_3 + 10\bar{T}_5\frac{\partial F^{(0)}_{(2,3)}}{\partial \bar{T}_1}) = 0 \tag{6.35}$$

After the change of variables $\bar{T} \to T$ and the identification $\widetilde{F}_{(2,3)} = \widetilde{F}_{(3,2)}$ described in s.6.3, one can check that the j-invariants of the curves coincide with each other:

$$j^{(2,3)} = j^{(3,2)} \tag{6.36}$$

i.e. the curves are isomorphic. At the critical points, the isomorphism is given just by

$$x^{(2,3)} \sim y^{(3,2)} \qquad y^{(2,3)} \sim x^{(3,2)} \tag{6.37}$$

Further details on equivalence of these Riemann surfaces can be found in Appendix B.
Note that this is again the off-KP duality.

6.5 Duality between $\rho^{(0|1)}$'s

The isomorphism between the elliptic curves $\mathcal{C}_{(3,2)} \xrightarrow{\phi} \mathcal{C}_{(2,3)}$ is given at the level of coordinate functions by the following linear (!) explicit expressions

$$\phi^* y^{(2,3)} = \frac{10}{3}T_5 \cdot x^{(3,2)} + \frac{4}{5}\frac{T_4}{T_5} \cdot y^{(3,2)} + 2\left(T_2 + \frac{32}{75}\frac{T_4^3}{T_5^2}\right) \tag{6.38}$$

$$\phi^* x^{(2,3)} = +\frac{1}{5}\frac{1}{T_5} \cdot y^{(3,2)} + \frac{3}{5}\frac{T_3}{T_5} \tag{6.39}$$

Using this, one can see that

$$\phi^* \left[\rho^{(0|1)}_{(2,3)}\right]_{\mathrm{mer}} = \left[\rho^{(0|1)}_{(3,2)}\right]_{\mathrm{mer}} \quad (\bmod\ exact\ form) \tag{6.40}$$

where we regard $\rho^{(0|1)} = \rho^{(0|1)}(x)dx$ as a 1-form, and $\left[\rho^{(0|1)}\right]_{\mathrm{mer}}$ is its meromorphic part (without the $W(x)$-term, which simply cancels the singular part of the expansion of y w.r.t. x nearby $x = \infty$ and

thus, given the local coordinate x, can be recovered easily). Therefore, the meromorphic differentials $y^{(3,2)}dx^{(3,2)}$ and $y^{(2,3)}dx^{(2,3)}$ differ by a meromorphic differential with zero periods. This can be also expressed as

$$\int\limits_{A_i} \rho^{(0|1)}_{(3,2)} = \int\limits_{\phi(A_i)} \rho^{(0|1)}_{(2,3)} \qquad\qquad \int\limits_{B_i} \rho^{(0|1)}_{(3,2)} = \int\limits_{\phi(B_i)} \rho^{(0|1)}_{(2,3)} \tag{6.41}$$

where A_i, B_i are the cycles on the $\mathcal{C}_{(3,2)}$ (we assume that the contours do not encircle the ramification points $x = 0$ and $x = \infty$ produced by $W(x)$-terms in $\rho^{(0|1)}$ in both cases).

6.6 On $\rho^{(0|2)}$ in (2,3) and (3,2) models

The isomorphism ϕ, of course, connects the holomorphic differentials on the curves:

$$\phi^* \frac{dx^{(2,3)}}{y^{(2,3)}} \sim \frac{dx^{(3,2)}}{\left(y^{(3,2)}\right)^2 + A} \tag{6.42}$$

For the sake of simplicity, in what follows we work at the critical point (only T_1 and T_5 are switched on). Then one has

$$A = -5T_1T_5 \qquad B = 125T_5^3x^2 + 75T_5^2\frac{\partial F}{\partial T_1} \tag{6.43}$$

$$\phi^* \frac{dx^{(2,3)}}{y^{(2,3)}} = -2\cdot 5\cdot T_5 \,\frac{dx^{(3,2)}}{\left(y^{(3,2)}\right)^2 + A} \tag{6.44}$$

In the $p = 2$ case, there is a decomposition of $\rho^{(0|2)}$

$$\rho^{(0|2)} = \rho^{(0|2)}_{hol} + \rho^{(0|2)}_{glob} - \rho^{(0|2)}_{loc} \tag{6.45}$$

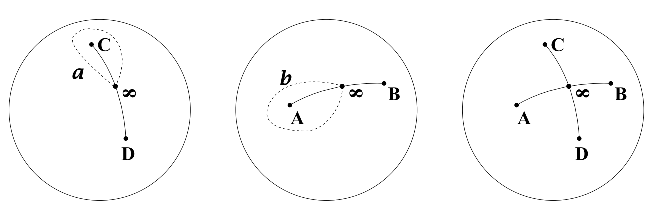
$$\rho^{(0|2)}_{(3,2),hol} = 5^4\,T_5^4\,\frac{\partial^2 F}{\partial T_1^2}\frac{dxdz}{(y^2(x)+A)(y^2(z)+A)} \tag{6.46}$$

$$\rho^{(0|2)}_{(3,2),glob} = \frac{dxdz}{4(x-z)^2(y^2(x)+A)(y^2(z)+A)}\Big[2\cdot 3\cdot 5^2\cdot T_1^2T_5^2 + 2\cdot 3\cdot 5\cdot T_1T_5\cdot y(x)y(z)+$$

$$-y(x)\left\{2^2 3^2 5^2 T_5^2\frac{\partial F}{\partial T_1} + 5^3T_5^3(z^2+2xz)\right\} - y(z)\left\{2^2 3^2 5^2 T_5^2\frac{\partial F}{\partial T_1} + 5^3T_5^3(x^2+2xz)\right\}\Big] \tag{6.47}$$

$$\rho^{(0|2)}_{(3,2),loc} = \frac{\left[x^{1/3}(2z+x) + z^{1/3}(2x+z)\right]dx\,dz}{4\,x^{2/3}\,z^{2/3}\,(x-z)^2} \tag{6.48}$$

Note that $\rho^{(0|2)}_{(3,2),glob}$, $\rho^{(0|2)}_{(3,2),hol}$, $\rho^{(0|2)}_{(3,2),loc}$ have the same defining properties as $\rho^{(0|2)}_{(2,3)}$ discussed in s.2.4. One can also try to formulate it in terms of the correlators (see s.2.5), however, here this is more difficult to find an appropriate local CFT. The scheme of cuts of the covering $\mathcal{C}_{(3,2)} \xrightarrow{\pi_{(3,2)}} \mathbb{CP}^1$ is



Here $[a]$ and $[b]$ are the standard homology basis on the torus. If one considers, say, the collection of fields (X_1, X_2, X_3) on the sheets, there is no linear combination of them which diagonalizes the monodromies around all points simultaneously.

For the (2,3) model one has

$$\rho^{(0|2)}_{(2,3),hol} = \frac{5^2}{2^2}\bar{T}_5^2 \frac{\partial^2 F}{\partial \bar{T}_1^2}\frac{dxdz}{y(x)y(z)} \tag{6.49}$$

$$\rho^{(0|2)}_{(2,3),glob} = \frac{dxdz}{(x-z)^2 y(x)y(z)}\left[\frac{1}{16}\left\{5^2\bar{T}_5^2 xz + 10\bar{T}_1\bar{T}_5\right\}(x+z) + 5\bar{T}_5\frac{\partial F}{\partial \bar{T}_1}\right] \tag{6.50}$$

$$\rho^{(0|2)}_{(2,3),loc} = \frac{[x+z]\,dx\,dz}{4\,x^{1/2}\,z^{1/2}\,(x-z)^2} \tag{6.51}$$

The singularity at $x = z$:

$$\rho^{(0|2)}_{(3,2),glob} \sim \rho^{(0|2)}_{(3,2),loc} \sim \frac{3}{2}\frac{dxdz}{(x-z)^2} \tag{6.52}$$

$$\rho^{(0|2)}_{(2,3),glob} \sim \rho^{(0|2)}_{(2,3),loc} \sim \frac{1}{2}\frac{dxdz}{(x-z)^2} \tag{6.53}$$

Both $\rho^{(0|2)}_{(2,3)}$ and $\rho^{(0|2)}_{(3,2)}$ can be constructed in terms of the coverings $\mathcal{C}_{(2,3)} \stackrel{\pi_{(2,3)}}{\longrightarrow} \mathbb{CP}^1$ and $\mathcal{C}_{(3,2)} \stackrel{\pi_{(3,2)}}{\longrightarrow} \mathbb{CP}^1$. The curves are isomorphic $\mathcal{C}_{(3,2)} \stackrel{\phi}{\to} \mathcal{C}_{(2,3)}$ but, of course, $\pi_{(3,2)} \neq \pi_{(2,3)}\phi$.

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A Derivation of (4.22)

Lemma A.1 *There exist the following formulae for* T_k *and* $\frac{\partial F^{(0)}}{\partial T_k}$:

$$T_k = -\frac{1}{k} res_{P=\infty}\left\{P\,ML^{-k/2}dP\right\} \tag{1.54}$$

$$\frac{\partial F^{(0)}}{\partial T_k} = -res_{P=\infty}\left\{P\,ML^{k/2}dP\right\} \tag{1.55}$$

Proof: Let us denote

$$H := \sum_{k=0}^{N}\frac{(2k+1)!!}{k!}T_{2k+1}\,u^k \tag{1.56}$$

Then, one can rewrite (4.20) as $\frac{\partial H}{\partial T_1} = 0$. From (4.2) and the dispersionless counterpart of (4.4) it follows that

$$\frac{\partial^2 F^{(0)}}{\partial T_1 \partial T_{2k+1}} = -\text{res} L^{k+1/2} = \frac{(2k+1)!!}{(k+1)!} u^{k+1} \tag{1.57}$$

$$\Rightarrow \quad \frac{\partial u}{\partial T_{2k+1}} = \frac{(2k+1)!!}{k!} u^k \frac{\partial u}{\partial T_1} \tag{1.58}$$

Then

$$\frac{\partial H}{\partial T_1} = 0 \Rightarrow \frac{\partial H}{\partial T_{2k+1}} = 0 \tag{1.59}$$

thus $H = 0$ (note that we have also checked in the way that this equation respects all the KdV-flows). One can see explicitly that the equation $H = 0$ can be represented as follows

$$T_1 = -\text{res} \left\{ P\, M L^{-1/2} dP \right\} \tag{1.60}$$

For $k > 0$ the equations

$$T_{2k+1} = -\frac{1}{2k+1} \text{res} \left\{ P\, M L^{-(2k+1)/2} dP \right\} \tag{1.61}$$

follow straightforwardly from formula (4.21) for M. Thus, it remains to verify that

$$\frac{\partial F^{(0)}}{\partial T_{2k+1}} = -\text{res} \left\{ P\, M L^{k+1/2} dP \right\} = \sum_{s=1}^{N} (-1)^s M_{s-1} u^{s+k-1} \frac{(2k+1)!! \cdot (2s-1)!!}{(s+k+1)!} \tag{1.62}$$

Differentiating it w.r.t. T_1 (note that, doing this, we lose nothing, because in the KdV hierarchy only the quantities containing, at least, first derivatives of F in T_1 make any sense) and taking into account that, from the string equation (4.20), one obtains

$$\frac{\partial M_{s-1}}{\partial T_1} = (2s+1) M_s \frac{\partial u}{\partial T_1} \qquad , \qquad M_0 \frac{\partial u}{\partial T_1} = -1 \tag{1.63}$$

one finally arrives at

$$\frac{\partial F^{(0)}}{\partial T_{2k+1} \partial T_1} = \frac{(2k+1)!!}{(k+1)!} u^{k+1} \tag{1.64}$$

which is the same as (1.57). $\square$

For reference, we also write down explicit formulae for T_k, M_s and $\frac{\partial F^{(0)}}{\partial T_k}$:

$$T_{2k+1} = \frac{1}{2k+1} \sum_{s=k}^{N} (-1)^{s-k} M_{s-1} u^{s-k} \frac{(2s-1)!!}{(2k-1)!! \cdot (s-k)!} \tag{1.65}$$

$$M_s = \sum_{k=1}^{N} T_{2k+1} \frac{(2k+1)!!}{(2s+1)!!(k-s-1)!} u^{k-s-1} \tag{1.66}$$

$$\frac{\partial F^{(0)}}{\partial T_{2k+1}} = \sum_{s=1}^{N} (-1)^s M_{s-1} u^{s+k-1} \frac{(2k+1)!! \cdot (2s-1)!!}{(s+k+1)!} \tag{1.67}$$

B On equivalence of Riemann surfaces

B.1 $y^2 = x^2 - 1$ and other hyperelliptic curves

The complex curve $y^2 = x^2 - 1$ is an ordinary Riemann sphere with coordinate z, where $x = \frac{1}{2}(z + 1/z)$ and $y = \frac{1}{2}(z - 1/z)$. There are no holomorphic differentials on it, and the kernel, the bi-differential with a double pole on the diagonal can be represented in a variety of ways:

$$B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} = \frac{1}{2}\frac{dx_1 dx_2}{(x_1 - x_2)^2}\left(1 + \frac{x_1 x_2 - 1}{y_1 y_2}\right) = \frac{1}{2}\frac{dy_1 dy_2}{(y_1 - y_2)^2}\left(1 + \frac{y_1 y_2 + 1}{x_1 x_2}\right) \tag{2.68}$$

For the generic hyperelliptic curve, $y^2 = P_n(x) = a_n x^n + \ldots + a_0$

$$B = \frac{1}{2}\frac{dx_1 dx_2}{(x_1 - x_2)^2}\left(1 + \frac{\tilde{P}_n(x_1, x_2)}{y_1 y_2}\right) \tag{2.69}$$

where the new polynomial $\tilde{P}_n$ is defined by the three conditions: $\tilde{P}_n(x, x) = P_n(x)$, symmetricity $\tilde{P}_n(x_1, x_2) = \tilde{P}_n(x_2, x_1)$ and restricted growth at infinity, $\tilde{P}_n(x_1, x_2) \sim x_1^{[(n+1)/2]}$. This means that for even $n = 2m$

$$\begin{gathered}\tilde{P}_{2m}(x_1, x_2) = a_{2m} x_1^m x_2^m + a_{2m-1}\frac{x_1^m x_2^{m-1} + x_1^{m-1} x_2^m}{2} + \\ + a_{2m-2}\Big(\alpha_{2m-2}\frac{x_1^m x_2^{m-2} + x_1^{m-2} x_2^m}{2} + (1 - \alpha_{2m-2}) x_1^{m-1} x_2^{m-1}\Big) + \ldots\end{gathered} \tag{2.70}$$

while for odd $n = 2m - 1$

$$\tilde{P}_{2m-1}(x_1, x_2) = a_{2m-1}\frac{x_1^m x_2^{m-1} + x_1^{m-1} x_2^m}{2} + \ldots$$

Extra $\frac{m(m-1)}{2}$ α-parameters describe possible bilinear combinations of holomorphic differentials which can be added to B, for example, α_{2m-2} is actually a coefficient in front of $x_1^{m-2} x_2^{m-2}(x_1 - x_2)^2$. This ambiguity can be fixed by specifying A-periods of the kernel B. In particular, the Bergmann kernel corresponds to the trivial A-periods

$$\oint_{A_i} B = 0 \tag{2.71}$$

B.2 $y^3 = x^2 - 1$

This curve is an ordinary torus with extended discrete $Z_2 \times Z_3$ symmetry. In the dual coordinates $X = y$, $Y = x$ it acquires the usual hyperelliptic form $Y^2 = X^3 + 1$, and the (un-normalized) holomorphic differential and the Bergmann kernel are

$$v = \frac{dX}{Y}, \quad B = \frac{1}{2}\frac{dX_1 dX_2}{(X_1 - X_2)^2}\left(1 + \frac{X_1^2 X_2 + X_1 X_2^2 + 2}{2Y_1 Y_2}\right) \tag{2.72}$$

In coordinates x, y one has instead a representation as a triple covering with the three ramification points at $x = \pm 1$ and $x = \infty$. The local coordinates in the vicinities of these points are

$$\begin{aligned} x &= \pm 1 + \xi^3, & y \sim \xi; dx \sim \xi^2 d\xi \;\Big| \\ x &= \frac{1}{\xi^3}, & y \sim \frac{1}{\xi^2}; dx \sim \frac{d\xi}{\xi^4} \;\Big| \end{aligned} \tag{2.73}$$

Accordingly, the holomorphic differential and the Bergmann kernel acquire the form

$$v = \frac{2}{3}\frac{dx}{y^2}, \quad B = \frac{dx_1 dx_2}{(x_1 - x_2)^2}\left(\frac{y_1^2 + y_1 y_2 + y_2^2}{3 y_1 y_2}\right)^2 - \frac{1}{9}\frac{(y_1 + y_2) dx_1 dx_2}{y_1^2 y_2^2} \tag{2.74}$$

In the first term in B the numerator vanishes when $y_2 = \pm e^{2\pi i/3} y_1$ and cancels the unwanted poles at the points $x_1 = x_2$ with $y_1 \neq y_2$. The second term serves to cancel poles at infinity.

The modified Begrmann kernel, with the pole at $x_2 = -x_1$ is

$$B^* = \frac{1}{2}\frac{dX_1 dX_2}{(X_1 - X_2)^2}\left(1 - \frac{X_1^2 X_2 + X_1 X_2^2 + 2}{2Y_1 Y_2}\right) = \\ = -\frac{dx_1 dx_2}{(x_1 + x_2)^2}\left(\frac{y_1^2 + y_1 y_2 + y_2^2}{3 y_1 y_2}\right)^2 + \frac{1}{9}\frac{(y_1 + y_2) dx_1 dx_2}{y_1^2 y_2^2} \tag{2.75}$$

B.3 $y^3 = A_1(x)y + C_2(x)$ from $Y^2 = P_3(X)$

Here $A_1(x) = a_{11}x + a_{10}$, $C_2(x) = c_{22}x^2 + c_{21}x + c_{20}$ and $P_3(X) = p_{33}X^3 + p_{32}X^2 + p_{31}X + p_{30}$.

The equation $y^3 = A_1(x)y + C_2(x)$ is quadratic in x, and the function $x(y)$ has the four order-two ramification points at $y = \infty$ and at the three roots of discriminant in the expression

$$x = \frac{-(c_{21} + a_{11}y) \pm \sqrt{(c_{21} + a_{11}y)^2 - 4c_{22}(c_{20} + a_{10}y - y^3)}}{2c_{22}} \tag{2.76}$$

At infinity

$$X \sim \frac{1}{\xi^2}, \quad Y \sim \frac{1}{\xi^3} \tag{2.77}$$

and

$$x \sim \frac{1}{\xi^3}, \quad y \sim \frac{1}{\xi^2} \tag{2.78}$$

Therefore,

$$Y = ux + vy + w, \quad X = py + q \tag{2.79}$$

The holomorphic differential $\frac{dX}{Y} \sim \frac{dy}{ux+vy+w}$ does not have poles at zeroes of the denominator, provided these zeroes are located exactly at the ramification points, where y has double zeroes. This means that actually

$$Y = ux + vy + w \sim 2c_{22}x + a_{11}y + c_{21} \tag{2.80}$$

Substituting this expression into $Y^2 = P_3(X)$ one obtains

$$\left(p_{33}p^3\right)y^3 + \left(3p_{33}p^2 q + p_{32}p^2 - a_{11}^2\right)y^2 + \left(3p_{33}pq^2 + 2p_{32}pq + p_{31}p - 2a_{11}c_{21}\right)y - 4a_{11}c_{22}xy = \\ = 4c_{22}^2 x^2 + 4c_{22}c_{21}x + c_{21}^2 - P_3(q)$$

which coincides with $y^3 = A_1(x)y + C_2(x)$ provided

$$\begin{cases} 2a_{11}^2 = p^2 P_3''(q) \\ p_{33}p^3 = 4c_{22} \\ P_3(q) = c_{21}^2 - 4c_{20}c_{22} \\ 2a_{11}c_{21} - 4a_{10}c_{22} = pP_3'(q) \end{cases}$$

These equations can be used to define p, q and $A_1(x)$ for given $P_3(x)$. The remaining freedom is $x \to \alpha x + \beta$ and it can be used to fix two coefficients in $C_2(x)$.

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